UNIT 2

Trigonometry

The simplest definition of trigonometry goes something like this:

Trigonometry is the study of angles and triangles. However, the study of trigonometry is really much more complex and comprehensive. In this unit alone, you will find trigonometric functions of angles, verify trigonometric identities, solve trigonometric equations, and graph trigonometric functions. Plus, you'll use vectors to solve parametric equations and to model motion. Trigonometry has applications in construction, geography, physics, acoustics, medicine, meteorology, and navigation, among other fields.

Unit 2 interNET Projects

THE CYBER CLASSROOM

From the Internet, a new form of classroom has emerged—one that exists only on the Web. Classes are now being taught exclusively over the Internet to students living across the globe. In addition, traditional classroom teachers are posting notes, lessons, and other information on web sites that can be accessed by their students. At the end of each chapter in Unit 2, you will search the Web for trigonometry learning resources.

CHAPTER 5 (page 339)

Does anybody out there know anything about trigonometry? Across the United States and the world, students are attending classes right in their own homes. What is it like to learn in this new environment? Use the Internet to find trigonometry lessons.
Math Connection: Find and compare trigonometry lessons from this textbook and the Internet. Then, select one topic from Chapter 5 and write a summary of this topic using the information you have gathered from both the textbook and the Internet.

CHAPTER 6 (page 417)

What is your sine? Mathematicians, scientists, and others share their work by means of the Internet. What are some applications of the sine or cosine function?
Math Connection: Use the Internet to find more applications of the sine or cosine function. Find data on the Internet that can be modeled by using a sine or cosine curve. Graph the data and the sine or cosine function that approximates it on the same axes.

CHAPTER 7 (page 481)

That's as clear as mud! Teachers using the Internet to deliver their courses need to provide the same clear instructions and examples to their students that they would in a traditional classroom.
Math Connection: Research the types of trigonometry sample problems given in Internet lessons. Design your own web page that includes two sample trigonometry problems.

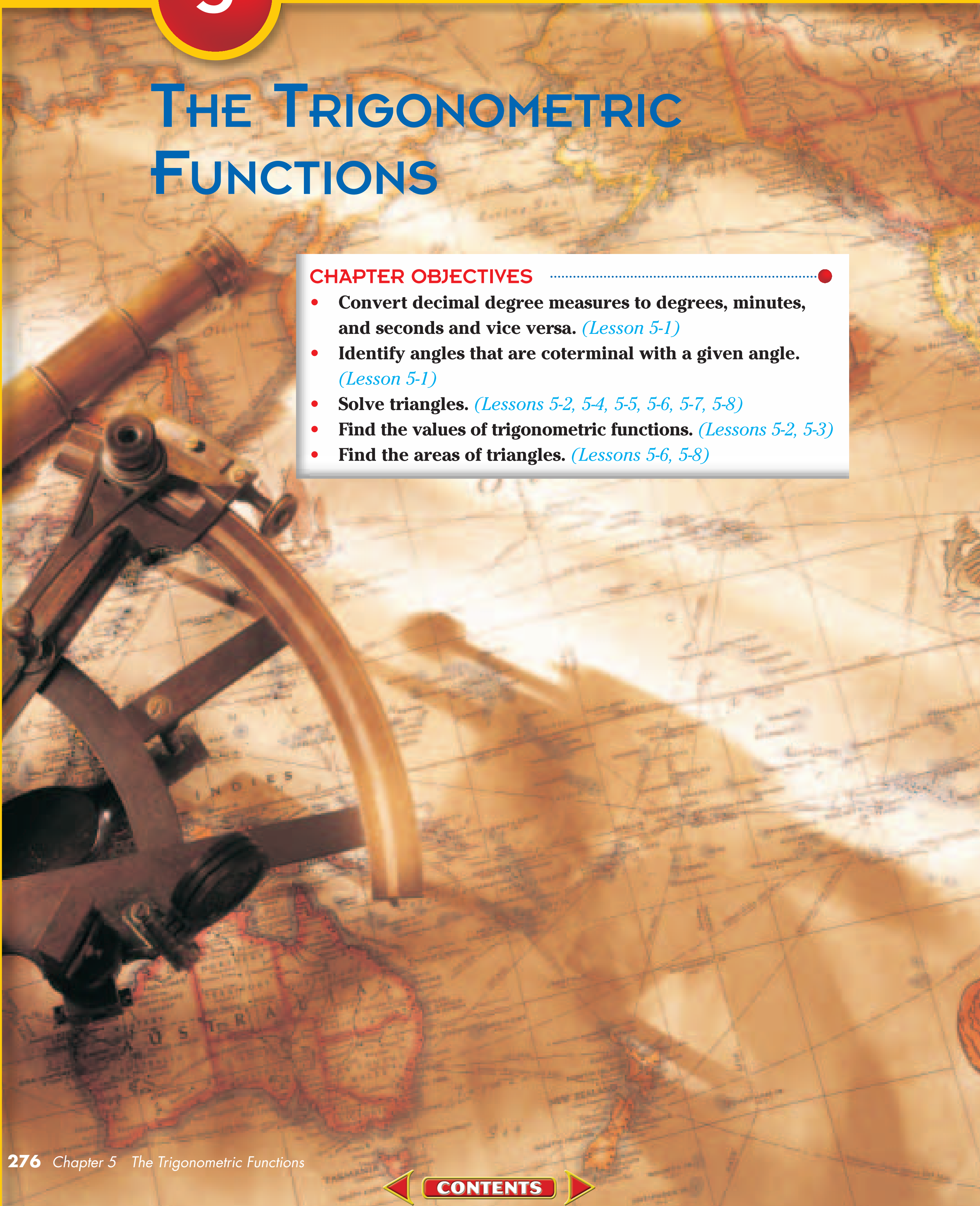
CHAPTER 8 (page 547)

Vivid Vectors Suppose you are taking a physics class that requires you to use vectors to represent real world situations. Can you find out more about these representations of direction and magnitude?
Math Connection: Research learning sites on the Internet to find more information about vector applications. Describe three real-world situations that can be modeled by vectors. Include vector diagrams of each situation.

interNET CONNECTION

For more information on the Unit Project, visit:
www.amc.glencoe.com

The Trigonometric Functions



CHAPTER OBJECTIVES

- **Convert decimal degree measures to degrees, minutes, and seconds and vice versa.** *(Lesson 5-1)*
- **Identify angles that are coterminal with a given angle.** *(Lesson 5-1)*
- **Solve triangles.** *(Lessons 5-2, 5-4, 5-5, 5-6, 5-7, 5-8)*
- **Find the values of trigonometric functions.** *(Lessons 5-2, 5-3)*
- **Find the areas of triangles.** *(Lessons 5-6, 5-8)*

5-1

Angles and Degree Measure

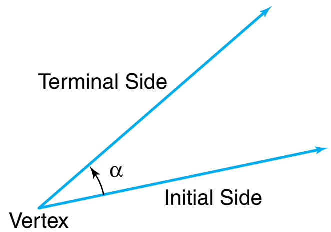
OBJECTIVES

- Convert decimal degree measures to degrees, minutes, and seconds and vice versa.
- Find the number of degrees in a given number of rotations.
- Identify angles that are coterminal with a given angle.

Real World Application

NAVIGATION The sextant is an optical instrument invented around 1730. It is used to measure the angular elevation of stars, so that a navigator can determine the ship's current latitude. Suppose a navigator determines a ship in the Pacific Ocean to be located at north latitude 15.735°. How can this be written as degrees, minutes, and seconds? *This problem will be solved in Example 1.*

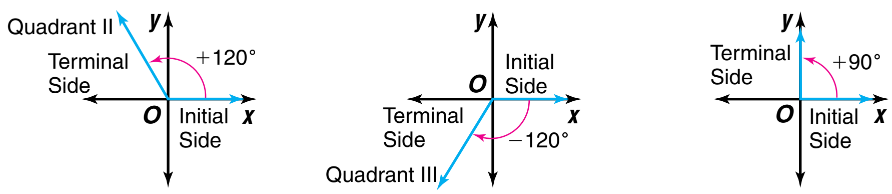
An angle may be generated by rotating one of two rays that share a fixed endpoint known as the **vertex.** One of the rays is fixed to form the **initial side** of the angle, and the second ray rotates to form the **terminal side.**



The measure of an angle provides us with information concerning the direction of the rotation and the amount of the rotation necessary to move from the initial side of the angle to the terminal side.

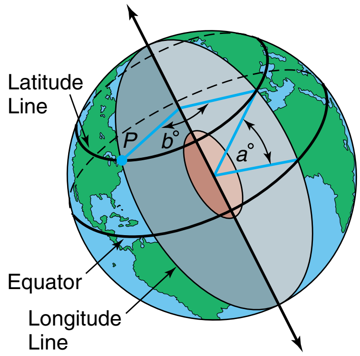
- If the rotation is in a counterclockwise direction, the angle formed is a *positive angle.*
- If the rotation is clockwise, it is a *negative angle.*

An angle with its vertex at the origin and its initial side along the positive x-axis is said to be in **standard position.** In the figures below, all of the angles are in standard position.



The most common unit used to measure angles is the **degree.** The concept of degree measurement is rooted in the ancient Babylonian culture. The Babylonians based their numeration system on 60 rather than 10 as we do today. In an equilateral triangle, they assigned the measure of each angle to be 60. Therefore, one sixtieth $\left(\frac{1}{60}\right)$ of the measure of the angle of an equilateral triangle was equivalent to one unit or degree (1°). The degree is subdivided into 60 equal parts known as **minutes** (1′), and the minute is subdivided into 60 equal parts known as **seconds** (1″).

Angles are used in a variety of real-world situations. For example, in order to locate every point on Earth, cartographers use a grid that contains circles through the poles, called *longitude lines,* and circles parallel to the equator, called *latitude lines.* Point P is located by traveling north from the equator through a central angle of $a°$ to a circle of latitude and then west along that circle through an angle of $b°$. Latitude and longitude can be expressed in degrees as a decimal value or in degrees, minutes, and seconds.



Example 1 Real World Application

NAVIGATION **Refer to the application at the beginning of the lesson.**

a. Change north latitude 15.735° to degrees, minutes, and seconds.

$15.735° = 15° + (0.735 \cdot 60)'$ — *Multiply the decimal portion of the degree measure by 60 to find the number of minutes.*

$= 15° + 44.1'$

$= 15° + 44' + (0.1 \cdot 60)''$ — *Multiply the decimal portion of the minute measure by 60 to find the number of seconds.*

$= 15° + 44' + 6''$

15.735° can be written as 15° 44′ 6″.

b. Write north latitude 39° 5′ 34″ as a decimal rounded to the nearest thousandth.

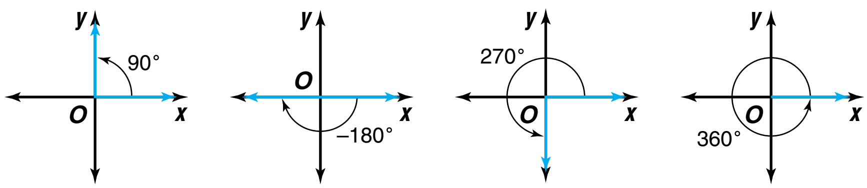
$39°\ 5'\ 34'' = 39° + 5'\left(\frac{1°}{60'}\right) + 34''\left(\frac{1°}{3600''}\right)$ or about 39.093°

39° 5′ 34″ can be written as 39.093°.

Graphing Calculator Tip

▶DMS on the [ANGLE] menu allows you to convert decimal degree values to degrees, minutes, and seconds.

If the terminal side of an angle that is in standard position coincides with one of the axes, the angle is called a **quadrantal angle.** In the figures below, all of the angles are quadrantal.



A full rotation around a circle is 360°. Measures of more than 360° represent multiple rotations.

Example 2 **Give the angle measure represented by each rotation.**

a. 5.5 rotations clockwise

$5.5 \times -360 = -1980$ — *Clockwise rotations have negative measures.*

The angle measure of 5.5 clockwise rotations is −1980°.

b. 3.3 rotations counterclockwise

$3.3 \times 360 = 1188$ — *Counterclockwise rotations have positive measures.*

The angle measure of 3.3 counterclockwise rotations is 1188°.

Two angles in standard position are called **coterminal angles** if they have the same terminal side. Since angles differing in degree measure by multiples of 360° are equivalent, every angle has infinitely many coterminal angles.

Coterminal Angles	If α is the degree measure of an angle, then all angles measuring $\alpha + 360k°$, where k is an integer, are coterminal with α.

The symbol α is the lowercase Greek letter alpha.

Any angle coterminal with an angle of 75° can be written as $75° + 360k°$, where k is the number of rotations around the circle. The value of k is a positive integer if the rotations are counterclockwise and a negative integer if the rotations are clockwise.

Examples

3 **Identify all angles that are coterminal with each angle. Then find one positive angle and one negative angle that are coterminal with the angle.**

a. 45°

All angles having a measure of $45° + 360k°$, where k is an integer, are coterminal with 45°. A positive angle is $45° + 360°(1)$ or 405°. A negative angle is $45° + 360°(-2)$ or $-675°$.

b. 225°

All angles having a measure of $225° + 360k°$, where k is an integer, are coterminal with 225°. A positive angle is $225° + 360(2)°$ or 945°. A negative angle is $225° + 360(-1)°$ or $-135°$.

4 **If each angle is in standard position, determine a coterminal angle that is between 0° and 360°. State the quadrant in which the terminal side lies.**

a. 775°

In $\alpha + 360k°$, you need to find the value of α. First, determine the number of complete rotations (k) by dividing 775 by 360.

$$\frac{775}{360} \approx 2.152777778$$

Then, determine the number of remaining degrees (α).

Method 1

$$\alpha \approx 0.152777778 \text{ rotations} \cdot 360°$$
$$\approx 55°$$

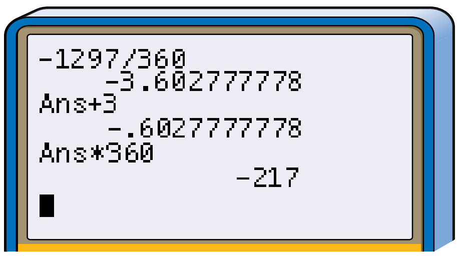
Method 2

$$\alpha + 360(2)° = 775°$$
$$\alpha + 720° = 775°$$
$$\alpha = 55°$$

The coterminal angle (α) is 55°. Its terminal side lies in the first quadrant.

b. −1297°

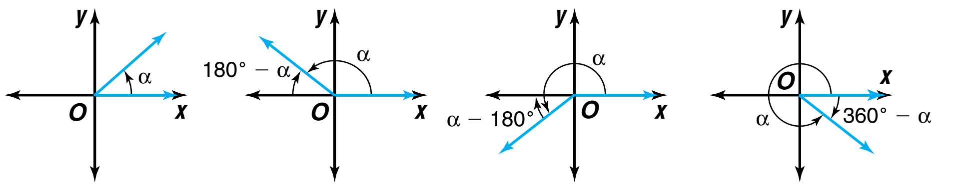
Use a calculator.



The angle is −217°, but the coterminal angle needs to be positive.
$360° - 217° = 143°$

The coterminal angle (α) is 143°. Its terminal side lies in the second quadrant.

If α is a nonquadrantal angle in standard position, its **reference angle** is defined as the acute angle formed by the terminal side of the given angle and the x-axis. You can use the figures and the rule below to find the reference angle for any angle α where $0° < \alpha < 360°$. If the measure of α is greater than 360° or less than 0°, it can be associated with a coterminal angle of positive measure between 0° and 360°.



Reference Angle Rule	For any angle α, $0° < \alpha < 360°$, its reference angle α' is defined by a. α, when the terminal side is in Quadrant I, b. $180° - \alpha$, when the terminal side is in Quadrant II, c. $\alpha - 180°$, when the terminal side is in Quadrant III, and d. $360° - \alpha$, when the terminal side is in Quadrant IV.

Example 5 **Find the measure of the reference angle for each angle.**

a. 120°

Since 120° is between 90° and 180°, the terminal side of the angle is in the second quadrant.

$180° - 120° = 60°$

The reference angle is 60°.

b. −135°

A coterminal angle of −135 is 360 − 135 or 225. Since 225 is between 180° and 270°, the terminal side of the angle is in the third quadrant.

$225° - 180° = 45°$

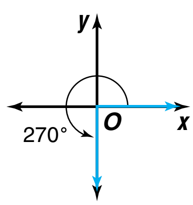
The reference angle is 45°.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the difference between an angle with a positive measure and an angle with a negative measure.
2. **Explain** how to write 29° 45′ 26″ as a decimal degree measure.
3. **Write** an expression for the measures of all angles that are coterminal with the angle shown.
4. **Sketch** an angle represented by 3.5 counterclockwise rotations. Give the angle measure represented by this rotation.



Guided Practice

Change each measure to degrees, minutes, and seconds.

5. 34.95°
6. −72.775°

Write each measure as a decimal to the nearest thousandth.

7. −128° 30′ 45″
8. 29° 6′ 6″

Give the angle measure represented by each rotation.

9. 2 rotations clockwise

10. 4.5 rotations counterclockwise

Identify all angles that are coterminal with each angle. Then find one positive angle and one negative angle that are coterminal with each angle.

11. $22°$

12. $-170°$

If each angle is in standard position, determine a coterminal angle that is between 0° and 360°. State the quadrant in which the terminal side lies.

13. $453°$

14. $-798°$

Find the measure of the reference angle for each angle.

15. $227°$

16. $-210°$

17. Geography Earth rotates once on its axis approximately every 24 hours. About how many degrees does a point on the equator travel through in one hour? in one minute? in one second?

EXERCISES

Practice

Change each measure to degrees, minutes, and seconds.

18. $-16.75°$

19. $168.35°$

20. $-183.47°$

21. $286.88°$

22. $27.465°$

23. $246.876°$

Write each measure as a decimal to the nearest thousandth.

24. $23° \, 14' \, 30''$

25. $-14° \, 5' \, 20''$

26. $233° \, 25' \, 15''$

27. $173° \, 24' \, 35''$

28. $-405° \, 16' \, 18''$

29. $1002° \, 30' \, 30''$

Give the angle measure represented by each rotation.

30. 3 rotations clockwise

31. 2 rotations counterclockwise

32. 1.5 rotations counterclockwise

33. 7.5 rotations clockwise

34. 2.25 rotations counterclockwise

35. 5.75 rotations clockwise

36. How many degrees are represented by 4 counterclockwise revolutions?

Identify all angles that are coterminal with each angle. Then find one positive angle and one negative angle that are coterminal with each angle.

37. $30°$

38. $-45°$

39. $113°$

40. $217°$

41. $-199°$

42. $-305°$

43. Determine the angle between 0° and 360° that is coterminal with all angles represented by $310° + 360k°$, where k is any integer.

44. Find the angle that is two counterclockwise rotations from 60°. Then find the angle that is three clockwise rotations from 60°.

If each angle is in standard position, determine a coterminal angle that is between 0° and 360°. State the quadrant in which the terminal side lies.

45. $400°$

46. $-280°$

47. $940°$

48. $1059°$

49. $-624°$

50. $-989°$

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51. In what quadrant is the terminal side of a $1275°$ angle located?

Find the measure of the reference angle for each angle.

52. $327°$ 53. $148°$ 54. $563°$ 55. $-420°$ 56. $-197°$ 57. $1045°$

58. Name four angles between $0°$ and $360°$ with a reference angle of $20°$.

Applications and Problem Solving

Real World Application

59. **Technology** A computer's hard disk is spinning at 12.5 revolutions per second. Through how many degrees does it travel in a second? in a minute?

60. **Critical Thinking** Write an expression that represents all quadrantal angles.

61. **Biking** During the winter, a competitive bike rider trains on a stationary bike. Her trainer wants her to warm up for 5 to 10 minutes by pedaling slowly. Then she is to increase the pace to 95 revolutions per minute for 30 seconds. Through how many degrees will a point on the outside of the tire travel during the 30 seconds of the faster pace?

62. **Flywheels** A high-performance composite flywheel rotor can spin anywhere between 30,000 and 100,000 revolutions per minute. What is the range of degrees through which the composite flywheel can travel in a minute? Write your answer in scientific notation.

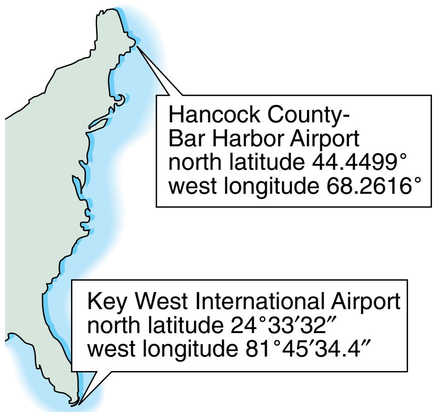
63. **Astronomy** On January 28, 1998, an x-ray satellite spotted a neutron star that spins at a rate of 62 times per second. Through how many degrees does this neutron star rotate in a second? in a minute? in an hour? in a day?

64. **Critical Thinking** Write an expression that represents any angle that is coterminal with a $25°$ angle, a $145°$ angle, and a $265°$ angle.

65. **Aviation** The locations of two airports are indicated on the map.

a. Write the latitude and longitude of the Hancock County–Bar Harbor airport in Bar Harbor, Maine, as degrees, minutes, and seconds.

b. Write the latitude and longitude of the Key West International Airport in Key West, Florida, as a decimal to the nearest thousandth.



66. **Entertainment** A tower restaurant in Sydney, Australia, is 300 meters above sea level and provides a $360°$ panoramic view of the city as it rotates every 70 minutes. A tower restaurant in San Antonio, Texas, is 750 feet tall. It revolves at a rate of one revolution per hour.

a. In a day, how many more revolutions does the restaurant in San Antonio make than the one in Sydney?

b. In a week, how many more degrees does a speck of dirt on the window of the restaurant in San Antonio revolve than a speck of dirt on the window of the restaurant in Sydney?

Mixed Review

interNET CONNECTION

Data Update For the latest information about motor vehicle production, visit our website at **www.amc.glencoe.com**

67. Manufacturing The percent of the motor vehicles produced in the United States since 1950 is depicted in the table at the right. *(Lesson 4-8)*

a. Write an equation to model the percent of the motor vehicles produced in the United States as a function of the number of years since 1950.

b. According to the equation, what percent of motor vehicles will be produced in the United States in the year 2010?

Motor Vehicle Production in the United States

Year	Percent
1950	75.7
1960	47.9
1970	28.2
1980	20.8
1990	20.1
1992	20.2
1993	23.3
1994	24.8
1997	22.7

Source: American Automobile Manufacturers Association

68. Solve $\sqrt[3]{6n + 5} - 15 = -10$. *(Lesson 4-7)*

69. Solve $\frac{x + 3}{x+2} = 2 - \frac{3}{x^2 + 5x + 6}$. *(Lesson 4-6)*

70. Use the Remainder Theorem to find the remainder if $x^3 + 8x + 1$ is divided by $x - 2$. *(Lesson 4-3)*

71. Write a polynomial equation of least degree with roots -5, -6, and 10. *(Lesson 4-1)*

72. If r varies inversely as t and $r = 18$ when $t = -3$, find r when $t = -11$. *(Lesson 3-8)*

73. Determine whether the graph of $y = \frac{x^2-1}{x + 1}$ has infinite discontinuity, jump discontinuity, or point discontinuity, or is continuous. Then graph the function. *(Lesson 3-7)*

74. Graph $f(x) = |(x + 1)^2 + 2|$. Determine the interval(s) for which the function is increasing and the interval(s) for which the function is decreasing. *(Lesson 3-5)*

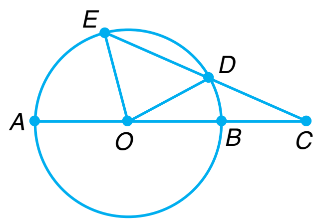
75. Use the graph of the parent function $f(x) = \frac{1}{x}$ to describe the graph of the related function $g(x) = \frac{3}{x} - 2$. *(Lesson 3-2)*

76. Solve the system of inequalities $y \leq 5$, $3y \geq 2x + 9$, and $-3y \leq 6x - 9$ by graphing. Name the coordinates of the vertices of the convex set. *(Lesson 2-6)*

77. Find $[f \cdot g](x)$ if $f(x) = x - 0.2x$ and $g(x) = x - 0.3x$. *(Lesson 1-2)*

78. SAT/ACT Practice $\overline{AB}$ is a diameter of circle O, and $m\angle BOD = 15°$. If $m\angle EOA = 85°$, find $m\angle ECA$.

A 85° **B** 50° **C** 70°
D 35° **E** 45°



Extra Practice See p. A34.

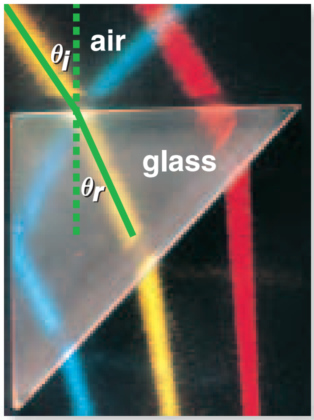
5-2 Trigonometric Ratios in Right Triangles

OBJECTIVE

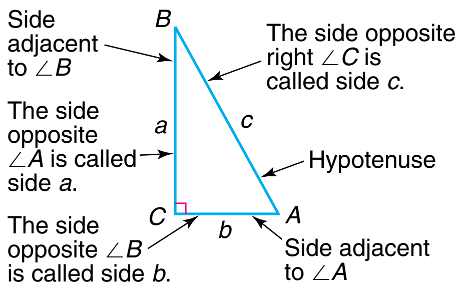
- Find the values of trigonometric ratios for acute angles of right triangles.

Real World Application

PHYSICS As light passes from one substance such as air to another substance such as glass, the light is bent. The relationship between the angle of incidence θ_i and the angle of refraction θ_r is given by Snell's Law, $\frac{\sin \theta_i}{\sin \theta_r} = n$, where $\sin \theta$ represents a trigonometric ratio and n is a constant called the *index of refraction*. Suppose a ray of light passes from air with an angle of incidence of 50° to glass with an angle of refraction of 32° 16′. Find the index of refraction of the glass. *This problem will be solved in Example 2.*

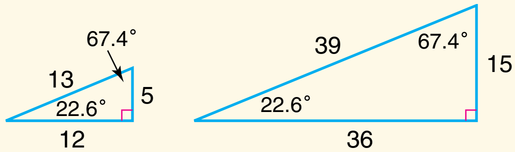


In a right triangle, one of the angles measures 90°, and the remaining two angles are *acute* and *complementary*. The longest side of a right triangle is known as the **hypotenuse** and is opposite the right angle. The other two sides are called **legs**. The leg that is a side of an acute angle is called the **side adjacent** to the angle. The other leg is the **side opposite** the angle.



GRAPHING CALCULATOR EXPLORATION

Use a graphing calculator to find each ratio for the 22.6° angle in each triangle. Record each ratio as a decimal. Make sure your calculator is in degree mode.



$R_1 = \frac{\text{side opposite}}{\text{hypotenuse}}$ $R_2 = \frac{\text{side adjacent}}{\text{hypotenuse}}$

$R_3 = \frac{\text{side opposite}}{\text{side adjacent}}$

Find the same ratios for the 67.4° angle in each triangle.

TRY THESE

1. Draw two other triangles that are similar to the given triangles.

2. Find each ratio for the 22.6° angle in each triangle.

3. Find each ratio for the 67.4° angle in each triangle.

WHAT DO YOU THINK?

4. Make a conjecture about R_1, R_2, and R_3 for any right triangle with a 22.6° angle.

5. Is your conjecture true for any 67.4° angle in a right triangle?

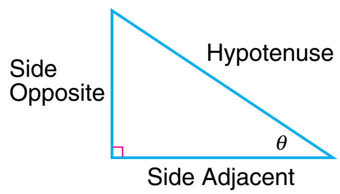
6. Do you think your conjecture is true for any acute angle of a right triangle? Why?

If two angles of a triangle are congruent to two angles of another triangle, the triangles are similar. If an acute angle of one right triangle is congruent to an acute angle of another right triangle, the triangles are similar, and the ratios of the corresponding sides are equal. Therefore, any two congruent angles of different right triangles will have equal ratios associated with them.

In right triangles, the Greek letter θ (theta) is often used to denote a particular angle.

The ratios of the sides of the right triangles can be used to define the **trigonometric ratios.** The ratio of the side opposite θ and the hypotenuse is known as the **sine.** The ratio of the side adjacent θ and the hypotenuse is known as the **cosine.** The ratio of the side opposite θ and the side adjacent θ is known as the **tangent.**

	Words	Symbol	Definition
Trigonometric Ratios	sine θ	$\sin\theta$	$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}$
	cosine θ	$\cos\theta$	$\cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$
	tangent θ	$\tan\theta$	$\tan\theta = \frac{\text{side opposite}}{\text{side adjacent}}$

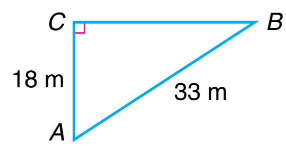


SOH-CAH-TOA is a mnemonic device commonly used for remembering these ratios.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Example 1

Find the values of the sine, cosine, and tangent for ∠B.



First, find the length of $\overline{BC}$

$(AC)^2 + (BC)^2 = (AB)^2$ *Pythagorean Theorem*

$18^2 + (BC)^2 = 33^2$ *Substitute 18 for AC and 33 for AB.*

$(BC)^2 = 765$

$BC = \sqrt{765}$ or $3\sqrt{85}$ *Take the square root of each side.* *Disregard the negative root.*

Then write each trigonometric ratio.

$\sin B = \frac{\text{side opposite}}{\text{hypotenuse}}$ $\qquad$ $\cos B = \frac{\text{side adjacent}}{\text{hypotenuse}}$ $\qquad$ $\tan B = \frac{\text{side opposite}}{\text{side adjacent}}$

$\sin B = \frac{18}{33}$ or $\frac{6}{11}$ $\qquad$ $\cos B = \frac{3\sqrt{85}}{33}$ or $\frac{\sqrt{85}}{11}$ $\qquad$ $\tan B = \frac{18}{3\sqrt{85}}$ or $\frac{6\sqrt{85}}{85}$

Trigonometric ratios are often simplified, but never written as mixed numbers.

In Example 1, you found the exact values of the sine, cosine, and tangent ratios. You can use a calculator to find the approximate decimal value of any of the trigonometric ratios for a given angle.

Example Real World Application

2 **PHYSICS** **Refer to the application at the beginning of the lesson. Find the index of refraction of the glass.**

$\frac{\sin \theta_i}{\sin \theta_r} = n$ *Snell's Law*

$\frac{\sin 50°}{\sin 32° 16'} = n$ *Substitute 50° for θ_i and 32° 16′ for θ_r.*

$\frac{0.7660444431}{0.5338605056} \approx n$ *Use a calculator to find each sine ratio.*

$1.434914992 \approx n$ *Use a calculator to find the quotient.*

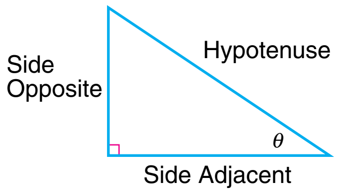
The index of refraction of the glass is about 1.4349.

Graphing Calculator Tip

If using your graphing calculator to do the calculation, make sure you are in degree mode.

In addition to the trigonometric ratios sine, cosine, and tangent, there are three other trigonometric ratios called **cosecant**, **secant**, and **cotangent**. These ratios are the reciprocals of sine, cosine, and tangent, respectively.

	Words	Symbol	Definition
Reciprocal Trigonometric Ratios	cosecant θ	csc θ	$\csc \theta = \frac{1}{\sin \theta}$ or $\frac{\text{hypotenuse}}{\text{side opposite}}$
	secant θ	sec θ	$\sec \theta = \frac{1}{\cos \theta}$ or $\frac{\text{hypotenuse}}{\text{side adjacent}}$
	cotangent θ	cot θ	$\cot \theta = \frac{1}{\tan \theta}$ or $\frac{\text{side adjacent}}{\text{side opposite}}$



These definitions are called the reciprocal identities.

Examples

3 **a. If $\cos \theta = \frac{3}{4}$, find $\sec \theta$.**

$\sec \theta = \frac{1}{\cos \theta}$

$\sec \theta = \frac{1}{\frac{3}{4}}$ or $\frac{4}{3}$

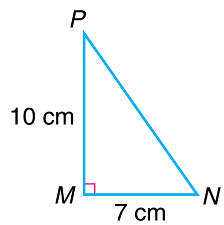
b. If $\csc \theta = 1.345$, find $\sin \theta$.

$\sin \theta = \frac{1}{\csc \theta}$

$\sin \theta = \frac{1}{1.345}$ or about 0.7435

4 **Find the values of the six trigonometric ratios for $\angle P$.**

First determine the length of the hypotenuse.



$(MP)^2 + (MN)^2 = (NP)^2$ *Pythagorean Theorem*

$10^2 + 7^2 = (NP)^2$ *Substitute 10 for MP and 7 for MN.*

$149 = (NP)^2$

$\sqrt{149} = NP$ *Disregard the negative root.*

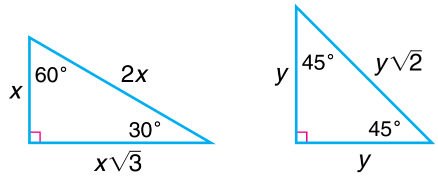
$\sin P = \frac{\text{side opposite}}{\text{hypotenuse}}$ $\quad \cos P = \frac{\text{side adjacent}}{\text{hypotenuse}}$ $\quad \tan P = \frac{\text{side opposite}}{\text{side adjacent}}$

$\sin P = \frac{7}{\sqrt{149}}$ or $\frac{7\sqrt{149}}{149}$ $\quad \cos P = \frac{10}{\sqrt{149}}$ or $\frac{10\sqrt{149}}{149}$ $\quad \tan P = \frac{7}{10}$

$\csc P = \frac{\text{hypotenuse}}{\text{side opposite}}$ $\quad \sec P = \frac{\text{hypotenuse}}{\text{side adjacent}}$ $\quad \cot P = \frac{\text{side adjacent}}{\text{side opposite}}$

$\csc P = \frac{\sqrt{149}}{7}$ $\quad \sec P = \frac{\sqrt{149}}{10}$ $\quad \cot P = \frac{10}{7}$

Consider the special relationships among the sides of $30°-60°-90°$ and $45°-45°-90°$ triangles.



These special relationships can be used to determine the trigonometric ratios for 30°, 45°, and 60°. You should memorize the sine, cosine, and tangent values for these angles.

θ	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Note that $\sin 30° = \cos 60°$ and $\cos 30° = \sin 60°$. This is an example showing that the sine and cosine are **cofunctions.** That is, if θ is an acute angle, $\sin \theta = \cos (90° - \theta)$. Similar relationships hold true for the other trigonometric ratios.

Cofunctions

$\sin \theta = \cos (90° - \theta)$ $\cos \theta = \sin (90° - \theta)$

$\tan \theta = \cot (90° - \theta)$ $\cot \theta = \tan (90° - \theta)$

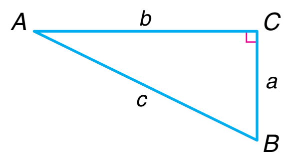
$\sec \theta = \csc (90° - \theta)$ $\csc \theta = \sec (90° - \theta)$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

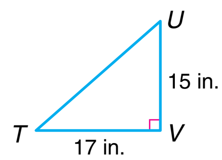
1. **Explain** in your own words how to decide which side is opposite the given acute angle of a right triangle and which side is adjacent to the given angle.
2. **State** the reciprocal ratios of sine, cosine, and tangent.
3. **Write** each trigonometric ratio for $\angle A$ in triangle *ABC*.



4. **Compare** sin *A* and cos *B*, csc *A* and sec *B*, and tan *A* and cot *B*.

Guided Practice

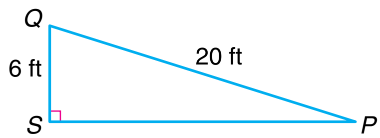
5. Find the values of the sine, cosine, and tangent for $\angle T$.



6. If $\sin \theta = \frac{2}{5}$, find $\csc \theta$.

7. If $\cot \theta = 1.5$, find $\tan \theta$.

8. Find the values of the six trigonometric ratios for $\angle P$.



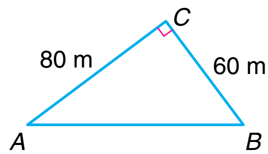
9. **Physics** You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose vertically polarized light with intensity I_o strikes a polarized filter with its axis at an angle of θ with the vertical. The intensity of the transmitted light I_t and θ are related by the equation $\cos \theta = \sqrt{\frac{I_t}{I_o}}$. If θ is 45°, write I_t as a function of I_o.

EXERCISES

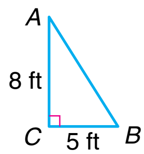
Practice

Find the values of the sine, cosine, and tangent for each $\angle A$.

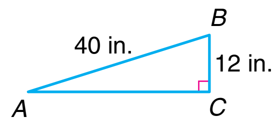
10.



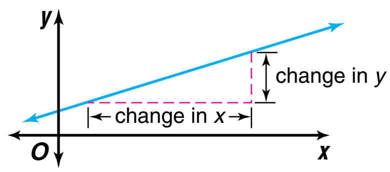
11.



12.



13. The slope of a line is the ratio of the change of y to the change of x. Name the trigonometric ratio of θ that equals the slope of line m.



14. If $\tan \theta = \frac{1}{3}$, find $\cot \theta$.

15. If $\sin \theta = \frac{3}{7}$, find $\csc \theta$.

16. If $\sec \theta = \frac{5}{9}$, find $\cos \theta$.

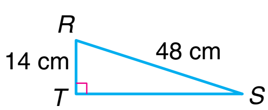
17. If $\csc \theta = 2.5$, find $\sin \theta$.

18. If $\cot \theta = 0.75$, find $\tan \theta$.

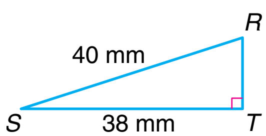
19. If $\cos \theta = 0.125$, find $\sec \theta$.

Find the values of the six trigonometric ratios for each $\angle R$.

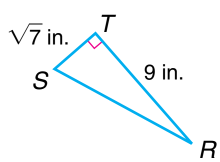
20.



21.



22.



23. If $\tan \theta = 1.3$, what is the value of $\cot (90° - \theta)$?

24. Use a calculator to determine the value of each trigonometric ratio.

a. sin 52° 47′ b. cos 79° 15′ c. tan 88° 22′ 45″

d. cot 36° (*Hint*: Tangent and cotangent have a reciprocal relationship.)

Graphing Calculator

25. Use the table function on a graphing calculator to complete the table. Round values to three decimal places.

θ	72°	74°	76°	78°	80°	82°	84°	86°	88°
sin	0.951	0.961							
cos	0.309								

a. What value does sin θ approach as θ approaches 90°?

b. What value does cos θ approach as θ approaches 90°?

26. Use the table function on a graphing calculator to complete the table. Round values to three decimal places.

θ	18°	16°	14°	12°	10°	8°	6°	4°	2°
sin	0.309	0.276							
cos	0.951								
tan									

a. What value does sin θ approach as θ approaches 0°?

b. What value does cos θ approach as θ approaches 0°?

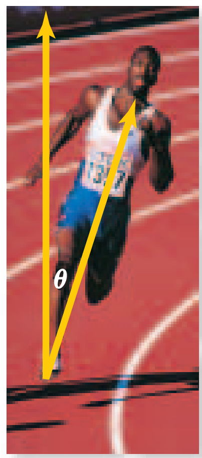
c. What value does tan θ approach as θ approaches 0°?

Applications and Problem Solving

27. **Physics** Suppose a ray of light passes from air to Lucite. The measure of the angle of incidence is 45°, and the measure of an angle of refraction is 27° 55′. Use Snell's Law, which is stated in the application at the beginning of the lesson, to find the index of refraction for Lucite.

28. **Critical Thinking** The sine of an acute $\angle R$ of a right triangle is $\frac{3}{7}$. Find the values of the other trigonometric ratios for this angle.

29. **Track** When rounding a curve, the acute angle θ that a runner's body makes with the vertical is called the angle of incline. It is described by the equation $\tan \theta = \frac{v^2}{gr}$, where v is the velocity of the runner, g is the acceleration due to gravity, and r is the radius of the track. The acceleration due to gravity is a constant 9.8 meters per second squared. Suppose the radius of the track is 15.5 meters.



a. What is the runner's velocity if the angle of incline is 11°?

b. Find the runner's velocity if the angle of incline is 13°.

c. What is the runner's velocity if the angle of incline is 15°?

d. Should a runner increase or decrease her velocity to increase his or her angle of incline?

30. **Critical Thinking** Use the fact that $\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}}$ and $\cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$ to write an expression for tan θ in terms of sin θ and cos θ.

CONTENTS

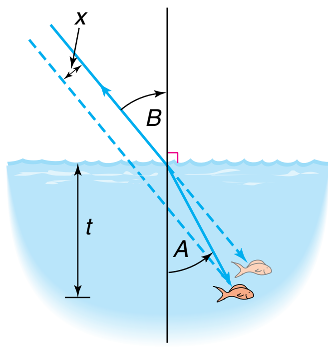
31. Architecture The angle of inclination of the sun affects the heating and cooling of buildings. The angle is greater in the summer than the winter. The sun's angle of inclination also varies according to the latitude. The sun's angle of inclination at noon equals $90° - L - 23.5° \times \cos\left[\frac{(N + 10)360}{365}\right]$. In this expression, L is the latitude of the building site, and N is the number of days elapsed in the year.

a. The latitude of Brownsville, Texas, is 26°. Find the angle of inclination for Brownsville on the first day of summer (day 172) and on the first day of winter (day 355).

b. The latitude of Nome, Alaska, is 64°. Find the angle of inclination for Nome on the first day of summer and on the first day of winter.

c. Which city has the greater change in the angle of inclination?

32. Biology An object under water is not exactly where it appears to be. The displacement x depends on the angle A at which the light strikes the surface of the water from below, the depth t of the object, and the angle B at which the light leaves the surface of the water. The measure of displacement is modeled by the equation $x = t\left(\frac{\sin (B - A)}{\cos A}\right)$. Suppose a biologist is trying to net a fish under water. Find the measure of displacement if t measures 10 centimeters, the measure of angle A is 41°, and the measure of angle B is 60°.



Mixed Review

33. Change 88.37° to degrees, minutes, and seconds. *(Lesson 5-1)*

34. Find the number of possible positive real zeros and the number of possible negative real zeros for $f(x) = x^4 + 2x^3 - 6x - 1$. *(Lesson 4-4)*

35. Business Luisa Diaz is planning to build a new factory for her business. She hires an analyst to gather data and develop a mathematical model. In the model $P(x) = 18 + 92x - 2x^2$, P is Ms. Diaz's monthly profit, and x is the number of employees needed to staff the new facility. *(Lesson 3-6)*

a. How many employees should she hire to maximize profits?

b. What is her maximum profit?

36. Find the value of $\begin{vmatrix} 7 & -3 & 5 \\ 4 & 0 & -1 \\ 8 & 2 & 0 \end{vmatrix}$. *(Lesson 2-5)*

37. Write the slope-intercept form of the equation of the line that passes through points at (2, 5) and (6, 3). *(Lesson 1-4)*

38. SAT/ACT Practice The area of a right triangle is 12 square inches. The ratio of the lengths of its legs is 2:3. Find the length of the hypotenuse.

A $\sqrt{13}$ in. **B** 26 in. **C** $2\sqrt{13}$ in. **D** 52 in. **E** $4\sqrt{13}$ in.

Extra Practice See p. A34.

5-3

Trigonometric Functions on the Unit Circle

OBJECTIVES

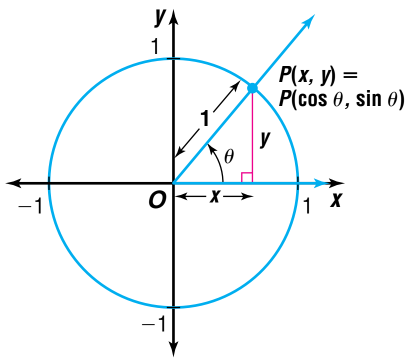
- Find the values of the six trigonometric functions using the unit circle.
- Find the values of the six trigonometric functions of an angle in standard position given a point on its terminal side.

Real World Application

FOOTBALL The longest punt in NFL history was 98 yards. The punt was made by Steve O'Neal of the New York Jets in 1969. When a football is punted, the angle made by the initial path of the ball and the ground affects both the height and the distance the ball will travel. If a football is punted from ground level, the maximum height it will reach is given by the formula $h = \frac{v_0^2 \sin^2 \theta}{2g}$, where v_0 is the initial velocity, θ is the measure of the angle between the ground and the initial path of the ball, and g is the acceleration due to gravity. The value of g is 9.8 meters per second squared. Suppose the initial velocity of the ball is 28 meters per second. Describe the possible maximum height of the ball if the angle is between 0° and 90°. *This problem will be solved in Example 2.*

A **unit circle** is a circle of radius 1. Consider a unit circle whose center is at the origin of a rectangular coordinate system. The unit circle is symmetric with respect to the x-axis, the y-axis, and the origin.

Consider an angle θ between 0° and 90° in standard position. Let $P(x, y)$ be the point of intersection of the angle's terminal side with the unit circle. If a perpendicular segment is drawn from point P to the x-axis, a right triangle is created. In the triangle, the side adjacent to angle θ is along the x-axis and has length x. The side opposite angle θ is the perpendicular segment and has length y. According to the Pythagorean Theorem, $x^2 + y^2 = 1$. We can find values for $\sin \theta$ and $\cos \theta$ using the definitions used in Lesson 5-2.



$$\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\sin \theta = \frac{y}{1} \text{ or } y \qquad \cos \theta = \frac{x}{1} \text{ or } x$$

Right triangles can also be formed for angles greater than 90°. In these cases, the reference angle is one of the acute angles. Similar results will occur. Thus, **sine** θ can be redefined as the y-coordinate and **cosine** θ can be redefined as the x-coordinate.

Sine and Cosine

If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$.

Since there is exactly one point $P(x, y)$ for any angle θ, the relations $\cos \theta = x$ and $\sin \theta = y$ are functions of θ. Because they are both defined using the unit circle, they are often called **circular functions.**

The domain of the sine and cosine functions is the set of real numbers, since $\sin \theta$ and $\cos \theta$ are defined for any angle θ. The range of the sine and the cosine functions is the set of real numbers between -1 and 1 inclusive, since $(\cos \theta, \sin \theta)$ are the coordinates of points on the unit circle.

In addition to the sine and cosine functions, the four other **trigonometric functions** can also be defined using the unit circle.

$$\tan \theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{y}{x} \qquad \csc \theta = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{1}{y}$$

$$\sec \theta = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{1}{x} \qquad \cot \theta = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{x}{y}$$

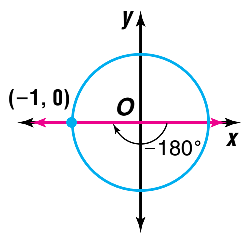
Since division by zero is undefined, there are several angle measures that are excluded from the domain of the tangent, cotangent, secant, and cosecant functions.

Examples

1 Use the unit circle to find each value.

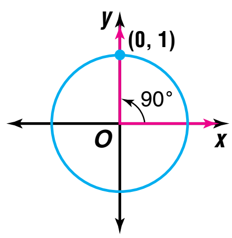
a. cos (−180°)

The terminal side of a $-180°$ angle in standard position is the negative x-axis, which intersects the unit circle at $(-1, 0)$. The x-coordinate of this ordered pair is $\cos(-180°)$. Therefore, $\cos(-180°) = -1$.



b. sec 90°

The terminal side of a 90° angle in standard position is the positive y-axis, which intersects the unit circle at $(0, 1)$. According to the definition of secant, $\sec 90° = \frac{1}{x}$ or $\frac{1}{0}$, which is undefined. Therefore, $\sec 90°$ is undefined.



Real World Application

2 FOOTBALL Refer to the application at the beginning of the lesson. Describe the possible maximum height of the ball if the angle is between 0° and 90°.

The expression $\sin^2 \theta$ means the square of the sine of θ or $(\sin \theta)^2$.

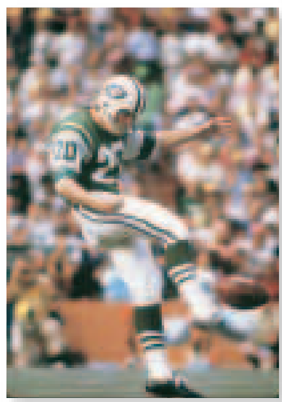
Find the value of h when $\theta = 0°$.

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$

$$h = \frac{28^2 \sin^2 0°}{2(9.8)} \qquad v_0 = 28, \theta = 0°, g = 9.8$$

$$h = \frac{28^2(0^2)}{2(9.8)} \qquad \sin 0° = 0$$

$$h = 0$$



Find the value of h when $\theta = 90°$.

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$

$$h = \frac{28^2 \sin^2 90°}{2(9.8)} \quad v_0 = 28,\ \theta = 90°,\ g = 9.8$$

$$h = \frac{28^2(1^2)}{2(9.8)} \quad \sin 90° = 1$$

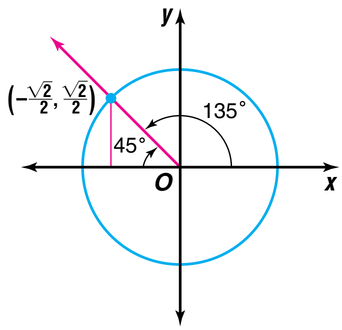
$$h = 40$$

The maximum height of the ball is between 0 meters and 40 meters.

The radius of a circle is defined as a positive value. Therefore, the signs of the six trigonometric functions are determined by the signs of the coordinates of x and y in each quadrant.

Example 3 **Use the unit circle to find the values of the six trigonometric functions for a 135° angle.**

Since 135° is between 90° and 180°, the terminal side is in the second quadrant. Therefore, the reference angle is 180° − 135° or 45°. The terminal side of a 45° angle intersects the unit circle at a point with coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Because the terminal side of a 135° angle is in the second quadrant, the x-coordinate is negative, and the y-coordinate is positive. The point of intersection has coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



$\sin 135° = y$

$\sin 135° = \frac{\sqrt{2}}{2}$

$\cos 135° = x$

$\cos 135° = -\frac{\sqrt{2}}{2}$

$\tan 135° = \frac{y}{x}$

$\tan 135° = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$

$\tan 135° = -1$

$\csc 135° = \frac{1}{y}$

$\csc 135° = \frac{1}{\frac{\sqrt{2}}{2}}$

$\csc 135° = \frac{2}{\sqrt{2}}$

$\csc 135° = \sqrt{2}$

$\sec 135° = \frac{1}{x}$

$\sec 135° = \frac{1}{-\frac{\sqrt{2}}{2}}$

$\sec 135° = -\frac{2}{\sqrt{2}}$

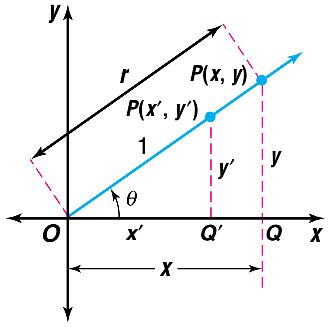
$\sec 135° = -\sqrt{2}$

$\cot 135° = \frac{x}{y}$

$\cot 135° = \frac{\frac{-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$\cot 135° = -1$

The sine and cosine functions of an angle in standard position may also be determined using the ordered pair of any point on its terminal side and the distance between that point and the origin.



Suppose $P(x, y)$ and $P'(x', y')$ are two points on the terminal side of an angle with measure θ, where P' is on the unit circle. Let $OP = r$. By the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$. Since P' is on the unit circle, $OP' = 1$. Triangles $OP'Q'$ and OPQ are similar. Thus, the lengths of corresponding sides are proportional.

$$\frac{x'}{1} = \frac{x}{r} \qquad \frac{y'}{1} = \frac{y}{r}$$

Therefore, $\cos\theta = x'$ or $\frac{x}{r}$ and $\sin\theta = y'$ or $\frac{y}{r}$.

All six trigonometric functions can be determined using x, y, and r. The ratios do not depend on the choice of P. They depend only on the measure of θ.

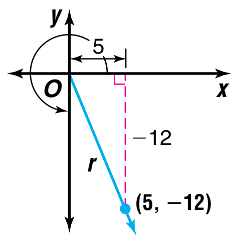
Trigonometric Functions of an Angle in Standard Position

For any angle in standard position with measure θ, a point $P(x, y)$ on its terminal side, and $r = \sqrt{x^2 + y^2}$, the trigonometric functions of θ are as follows.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

Example 4 **Find the values of the six trigonometric functions for angle θ in standard position if a point with coordinates $(5, -12)$ lies on its terminal side.**



You know that $x = 5$ and $y = -12$. You need to find r.

$r = \sqrt{x^2 + y^2}$ *Pythagorean Theorem*

$r = \sqrt{5^2 + (-12)^2}$ *Substitute 5 for x and −12 for y.*

$r = \sqrt{169}$ or 13 *Disregard the negative root.*

Now write the ratios.

$\sin\theta = \frac{y}{r}$ $\quad$ $\cos\theta = \frac{x}{r}$ $\quad$ $\tan\theta = \frac{y}{x}$

$\sin\theta = \frac{-12}{13}$ or $-\frac{12}{13}$ $\quad$ $\cos\theta = \frac{5}{13}$ $\quad$ $\tan\theta = \frac{-12}{5}$ or $-\frac{12}{5}$

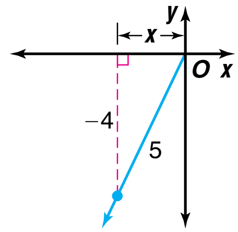
$\csc\theta = \frac{r}{y}$ $\quad$ $\sec\theta = \frac{r}{x}$ $\quad$ $\cot\theta = \frac{x}{y}$

$\csc\theta = \frac{13}{-12}$ or $-\frac{13}{12}$ $\quad$ $\sec\theta = \frac{13}{5}$ $\quad$ $\cot\theta = \frac{5}{-12}$ or $-\frac{5}{12}$

If you know the value of one of the trigonometric functions and the quadrant in which the terminal side of θ lies, you can find the values of the remaining five functions.

Example 5 **Suppose θ is an angle in standard position whose terminal side lies in Quadrant III. If $\sin\theta = -\frac{4}{5}$, find the values of the remaining five trigonometric functions of θ.**

To find the other function values, you must find the coordinates of a point on the terminal side of θ. Since $\sin\theta = -\frac{4}{5}$ and r is always positive, $r = 5$ and $y = -4$.



Find x.

$r^2 = x^2 + y^2$ *Pythagorean Theorem*

$5^2 = x^2 + (-4)^2$ *Substitute 5 for r and −4 for y.*

$9 = x^2$

$\pm 3 = x$ *Take the square root of each side.*

Since the terminal side of θ lies in Quadrant III, x must be negative. Thus, $x = -3$.

Now use the values of x, y, and r to find the remaining five trigonometric functions of θ.

$\cos\theta = \frac{x}{r}$

$\cos\theta = \frac{-3}{5}$ or $-\frac{3}{5}$

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{-4}{-3}$ or $\frac{4}{3}$

$\csc\theta = \frac{r}{y}$

$\csc\theta = \frac{5}{-4}$ or $-\frac{5}{4}$

$\sec\theta = \frac{r}{x}$

$\sec\theta = \frac{5}{-3}$ or $-\frac{5}{3}$

$\cot\theta = \frac{x}{y}$

$\cot\theta = \frac{-3}{-4}$ or $\frac{3}{4}$

Notice that the cosine and secant have the same sign. This will always be true since $\sec\theta = \frac{1}{\cos\theta}$. Similar relationships exist for the other reciprocal identities. *You will complete a chart for this in Exercise 4.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why csc 180° is undefined.
2. **Show** that the value of sin θ increases as θ goes from 0° to 90° and then decreases as θ goes from 90° to 180°.
3. **Confirm** that $\cot \theta = \frac{\cos \theta}{\sin \theta}$.
4. *Math Journal* **Draw** a unit circle. Use the drawing to complete the chart below that indicates the sign of the trigonometric functions in each quadrant.

Function	Quadrant			
	I	II	III	IV
sin α or csc α	+			
cos α or sec α	+			
tan α or cot α				

Guided Practice

Use the unit circle to find each value.

5. tan 180°
6. sec (−90°)

Use the unit circle to find the values of the six trigonometric functions for each angle.

7. 30°
8. 225°

Find the values of the six trigonometric functions for angle θ in standard position if a point with the given coordinates lies on its terminal side.

9. (3, 4)
10. (−6, 6)

Suppose θ is an angle in standard position whose terminal side lies in the given quadrant. For each function, find the values of the remaining five trigonometric functions for θ.

11. $\tan \theta = -1$; Quadrant IV
12. $\cos \theta = -\frac{1}{2}$; Quadrant II
13. **Map Skills** The distance around Earth along a given latitude can be found using the formula $C = 2\pi r \cos L$, where r is the radius of Earth and L is the latitude. The radius of Earth is approximately 3960 miles. Describe the distances along the latitudes as you go from 0° at the equator to 90° at the poles.

EXERCISES

Practice

Use the unit circle to find each value.

14. sin 90°
15. tan 360°
16. cot (−180°)
17. csc 270°
18. cos (−270°)
19. sec 180°
20. Find two values of θ for which $\sin \theta = 0$.
21. If $\cos \theta = 0$, what is $\sec \theta$?

www.amc.glencoe.com/self_check_quiz

Use the unit circle to find the values of the six trigonometric functions for each angle.

22. $45°$ **23.** $150°$ **24.** $315°$ **25.** $210°$ **26.** $330°$ **27.** $420°$

28. Find $\cot(-45°)$. **29.** Find $\csc 390°$.

Find the values of the six trigonometric functions for angle θ in standard position if a point with the given coordinates lies on its terminal side.

30. $(-4, -3)$ **31.** $(-6, 6)$ **32.** $(2, 0)$ **33.** $(1, -8)$ **34.** $(5, -3)$ **35.** $(-8, 15)$

36. The terminal side of one angle in standard position contains the point with coordinates $(5, -6)$. The terminal side of another angle in standard position contains the point with coordinates $(-5, 6)$. Compare the sines of these angles.

37. If $\sin\theta < 0$, where would the terminal side of the angle be located?

Suppose θ is an angle in standard position whose terminal side lies in the given quadrant. For each function, find the values of the remaining five trigonometric functions for θ.

38. $\cos\theta = -\frac{12}{13}$; Quadrant III **39.** $\csc\theta = 2$; Quadrant II

40. $\sin\theta = -\frac{1}{5}$; Quadrant IV **41.** $\tan\theta = 2$; Quadrant I

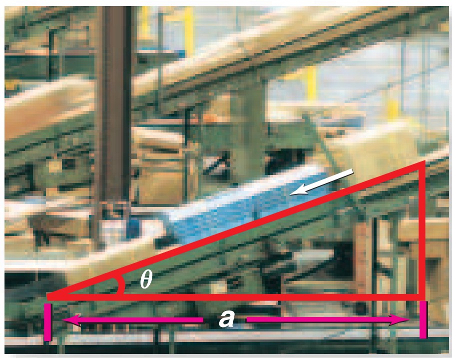
42. $\sec\theta = \sqrt{3}$; Quadrant IV **43.** $\cot\theta = 1$; Quadrant III

44. If $\csc\theta = -2$ and θ lies in Quadrant III, find $\tan\theta$.

Applications and Problem Solving

Real World Application

45. Physics If you ignore friction, the amount of time required for a box to slide down an inclined plane is $\sqrt{\frac{2a}{g\sin\theta\cos\theta}}$, where a is the horizontal distance defined by the inclined plane, g is the acceleration due to gravity, and θ is the angle of the inclined plane. For what values of θ is the expression undefined?



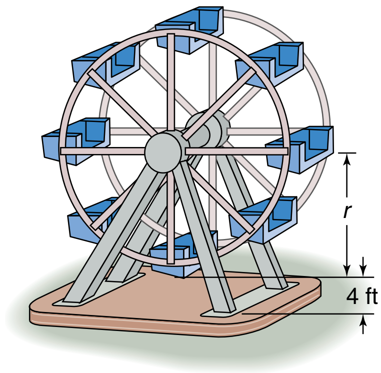
46. Critical Thinking For each statement, describe k.

a. $\tan(k \cdot 90°) = 0$ b. $\sec(k \cdot 90°)$ is undefined.

47. Physics For polarized light, $\cos\theta = \sqrt{\frac{I_t}{I_o}}$, where θ is the angle of the axis of the polarized filter with the vertical, I_t is the intensity of the transmitted light, and I_O is the intensity of the vertically-polarized light striking the filter. Under what conditions would $I_t = I_O$?

48. Critical Thinking The terminal side of an angle θ in standard position coincides with the line $y = -3x$ and lies in Quadrant II. Find the six trigonometric functions of θ.

49. Entertainment Domingo decides to ride the Ferris wheel at the local carnival. When he gets into the seat that is at the bottom of the Ferris wheel, he is 4 feet above the ground.



a. If the radius of the Ferris wheel is 36 feet, how far above the ground will Domingo be when his seat reaches the top?

b. The Ferris wheel rotates 300° counterclockwise and stops to let another passenger on the ride. How far above the ground is Domingo when the Ferris wheel stops?

c. Suppose the radius of the Ferris wheel is only 30 feet. How far above the ground is Domingo after the Ferris wheel rotates 300°?

d. Suppose the radius of the Ferris wheel is r. Write an expression for the distance from the ground to Domingo after the Ferris wheel rotates 300°.

Mixed Review

50. If $\csc \theta = \frac{7}{5}$, find $\sin \theta$. *(Lesson 5-2)*

51. If a $-840°$ angle is in standard position, determine a coterminal angle that is between 0° and 360° and state the quadrant in which the terminal side lies. *(Lesson 5-1)*

52. Solve $5 - \sqrt{b + 2} = 0$. *(Lesson 4-7)*

53. Solve $4x^2 - 9x + 5 = 0$ by using the quadratic formula. *(Lesson 4-2)*

54. If y varies directly as x and $y = 9$ when x is -15, find y when $x = 21$. *(Lesson 3-8)*

55. Graph the inverse of $f(x) = x^2 - 16$. *(Lesson 3-4)*

56. Find the multiplicative inverse of $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$. *(Lesson 2-5)*

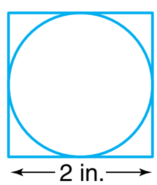
57. Solve the system of equations. *(Lesson 2-2)*

$8m - 3n - 4p = 6$
$4m + 9n - 2p = -4$
$6m + 12n + 5p = -1$

58. State whether each of the points at (9, 3), (−1, 2), and (2, −2) satisfy the inequality $2x - 4y \leq 7$. *(Lesson 1-8)*

59. Manufacturing The length of a nail is $2\frac{1}{2}$ inches. The manufacturer randomly measures the nails to test if their equipment is working properly. If the discrepancy is more than $\frac{1}{8}$ inch, adjustments must be made. Identify the type of function that models this situation. Then write a function for the situation. *(Lesson 1-7)*

60. SAT/ACT Practice In the figure at the right, the largest possible circle is cut out of a square piece of tin. What is the approximate total area of the remaining pieces of tin?



A 0.13 in^2 **B** 0.75 in^2 **C** 0.86 in^2
D 1.0 in^2 **E** 3.14 in^2

Extra Practice See p. A34.

5-4 Applying Trigonometric Functions

OBJECTIVE

- Use trigonometry to find the measures of the sides of right triangles.

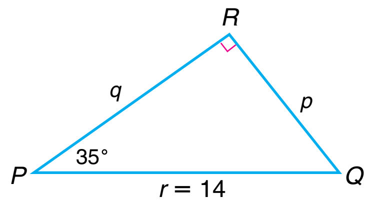
Real World Application

ENTERTAINMENT The circus has arrived and the roustabouts must put up the main tent in a field near town. A tab is located on the side of the tent 40 feet above the ground. A rope is tied to the tent at this point and then the rope is placed around a stake on the ground. If the angle that the rope makes with the level ground is 50° 15′, how long is the rope? What is the distance between the bottom of the tent and the stake? *This problem will be solved in Example 2.*

Trigonometric functions can be used to solve problems involving right triangles. The most common functions used are the sine, cosine, and tangent.

Examples

1 **If $P = 35°$ and $r = 14$, find q.**



From the art at the right, you know the measures of an angle and the hypotenuse. You want to know the measure of the side adjacent to the given angle. The cosine function relates the side adjacent to the angle and the hypotenuse.

$\cos P = \frac{q}{r}$ *$\cos = \frac{\text{side adjacent}}{\text{hypotenuse}}$*

$\cos 35° = \frac{q}{14}$ *Substitute 35° for P and 14 for r.*

$14 \cos 35° = q$ *Multiply each side by 14.*

$11.46812862 \approx q$ *Use a calculator.*

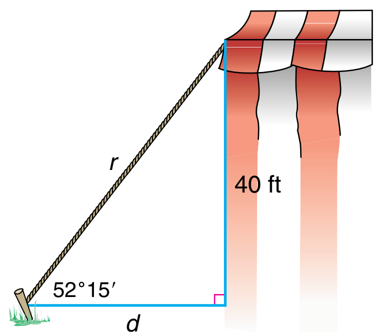
Therefore, q is about 11.5.

2 **ENTERTAINMENT** **Refer to the application above.**

Real World Application

a. If the angle that the rope makes with the level ground is 52° 15′, how long is the rope?

b. What is the distance between the bottom of the tent and the stake?



a. You know the measures of an angle and the side opposite the angle. To find the length of the rope, you need to know the measure of the hypotenuse. In this case, use the sine function.

(continued on the next page)

$$\sin 52° 15' = \frac{40}{r} \qquad \textit{sin} = \frac{\textit{side opposite}}{\textit{hypotenuse}}$$

$$r \sin 52° 15' = 40 \qquad \textit{Multiply each side by r.}$$

$$r = \frac{40}{\sin 52° 15'} \qquad \textit{Divide each side by sin 52° 15'.}$$

$$r \approx 50.58875357 \qquad \textit{Use a calculator.}$$

The rope is about 50.6 feet long.

b. To find the distance between the bottom of the tent and the stake, you need to know the length of the side adjacent to the known angle. Use the tangent function.

$$\tan 52° 15' = \frac{40}{d} \qquad \textit{tan} = \frac{\textit{side opposite}}{\textit{side adjacent}}$$

$$d \tan 52° 15' = 40 \qquad \textit{Multiply each side by d.}$$

$$d = \frac{40}{\tan 52° 15'} \qquad \textit{Divide each side by tan 52° 15'.}$$

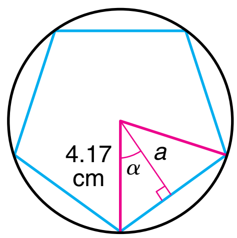
$$d \approx 30.97130911 \qquad \textit{Use a calculator.}$$

The distance between the bottom of the tent and the stake is about 31.0 feet.

You can use right triangle trigonometry to solve problems involving other geometric figures.

Example 3 **GEOMETRY** **A regular pentagon is inscribed in a circle with diameter 8.34 centimeters. The *apothem* of a regular polygon is the measure of a line segment from the center of the polygon to the midpoint of one of its sides. Find the apothem of the pentagon.**

First, draw a diagram. If the diameter of the circle is 8.34 centimeters, the radius is 8.34 ÷ 2 or 4.17 centimeters. The measure of α is 360° ÷ 10 or 36°.

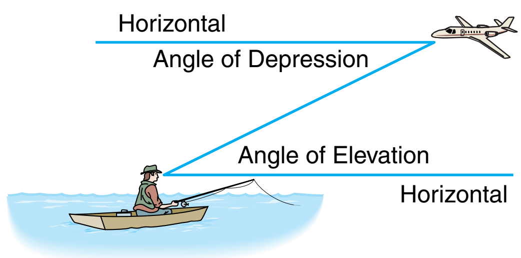


$$\cos 36° = \frac{a}{4.17} \qquad \textit{cos} = \frac{\textit{side adjacent}}{\textit{hypotenuse}}$$

$$4.17 \cos 36° = a \qquad \textit{Multiply each side by 4.17.}$$

$$3.373600867 \approx a \qquad \textit{Use a calculator.}$$

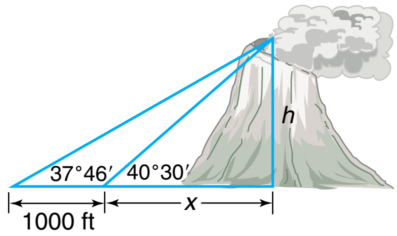
The apothem is about 3.37 centimeters.



There are many other applications that require trigonometric solutions. For example, surveyors use special instruments to find the measures of **angles of elevation** and **angles of depression.** An angle of elevation is the angle between a horizontal line and the line of sight from an observer to an object at a higher level. An angle of depression is the angle between a horizontal line and the line of sight from the observer to an object at a lower level. The angle of elevation and the angle of depression are equal in measure because they are alternate interior angles.

Example **4** **SURVEYING** **On May 18, 1980, Mount Saint Helens, a volcano in Washington, erupted with such force that the top of the mountain was blown off. To determine the new height at the summit of Mount Saint Helens, a surveyor measured the angle of elevation to the top of the volcano to be 37° 46′. The surveyor then moved 1000 feet closer to the volcano and measured the angle of elevation to be 40° 30′. Determine the new height of Mount Saint Helens.**

Draw a diagram to model the situation. Let h represent the height of the volcano and x represent the distance from the surveyor's second position to the center of the base of the volcano. Write two equations involving the tangent function.



$$\tan 37°\,46' = \frac{h}{1000 + x}$$
$$(1000 + x)\tan 37°\,46' = h$$
$$\tan 40°\,30' = \frac{h}{x}$$
$$x \tan 40°\,30' = h$$

Therefore, $(1000 + x)\tan 37°\,46' = x \tan 40°\,30'$. Solve this equation for x.

$$(1000 + x)\tan 37°\,46' = x \tan 40°\,30'$$
$$1000 \tan 37°\,46' + x \tan 37°\,46' = x \tan 40°\,30'$$
$$1000 \tan 37°\,46' = x \tan 40°\,30' - x \tan 37°\,46'$$
$$1000 \tan 37°\,46' = x(\tan 40°\,30' - \tan 37°\,46')$$
$$\frac{1000 \tan 37°\,46'}{\tan 40°\,30' - \tan 37°\,46'} = x$$
$$9765.826092 \approx x \quad \textit{Use a calculator.}$$

Use this value for x and the equation $x \tan 40°\,30' = h$ to find the height of the volcano.

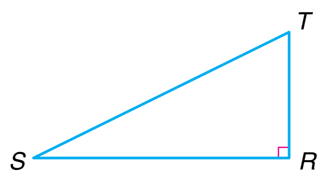
$$x \tan 40°\,30' = h$$
$$9765.826092 \tan 40°\,30' \approx h$$
$$8340.803443 \approx h \quad \textit{Use a calculator.}$$

The new height of Mount Saint Helens is about 8341 feet.

CHECK FOR UNDERSTANDING

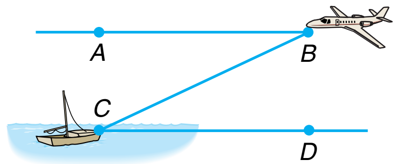
Communicating Mathematics

Read and study the lesson to answer each question.



1. **State** which trigonometric function you would use to solve each problem.
   a. If $S = 42°$ and $ST = 8$, find RS.
   b. If $T = 55°$ and $RT = 5$, find RS.
   c. If $S = 27°$ and $TR = 7$, find TS.
2. **Write** a problem that could be solved using the tangent function.

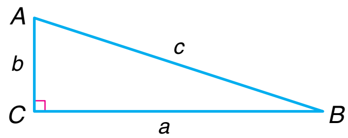
3. **Name** the angle of elevation and the angle of depression in the figure at the right. Compare the measures of these angles. Explain.



4. **Describe** a way to use trigonometry to determine the height of the building where you live.

Guided Practice

Solve each problem. Round to the nearest tenth.

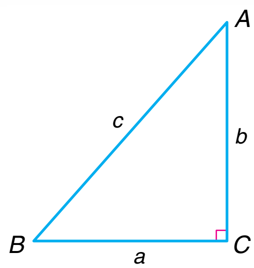


5. If $b = 13$ and $A = 76°$, find a.
6. If $B = 26°$ and $b = 18$, find c.
7. If $B = 16° \, 45'$ and $c = 13$, find a.
8. **Geometry** Each base angle of an isosceles triangle measures $55° \, 30'$. Each of the congruent sides is 10 centimeters long.
   a. Find the altitude of the triangle.
   b. What is the length of the base?
   c. Find the area of the triangle.
9. **Boating** The Ponce de Leon lighthouse in St. Augustine, Florida, is the second tallest brick tower in the United States. It was built in 1887 and rises 175 feet above sea level. How far from the shore is a motorboat if the angle of depression from the top of the lighthouse is $13° \, 15'$?

EXERCISES

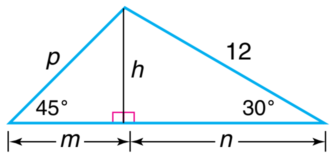
Practice

Solve each problem. Round to the nearest tenth.



Exercises 10–18

10. If $A = 37°$ and $b = 6$, find a.
11. If $c = 16$ and $B = 67°$, find a.
12. If $B = 62°$ and $c = 24$, find b.
13. If $A = 29°$ and $a = 4.6$, find c.
14. If $a = 17.3$ and $B = 77°$, find c.
15. If $b = 33.2$ and $B = 61°$, find a.
16. If $B = 49° \, 13'$ and $b = 10$, find a.
17. If $A = 16° \, 55'$ and $c = 13.7$, find a.
18. If $a = 22.3$ and $B = 47° \, 18'$, find c.
19. Find h, n, m, and p. Round to the nearest tenth.



Exercise 19

20. **Geometry** The apothem of a regular pentagon is 10.8 centimeters.
    a. Find the radius of the circumscribed circle.
    b. What is the length of a side of the pentagon?
    c. Find the perimeter of the pentagon.
21. **Geometry** Each base angle of an isosceles triangle measures $42° \, 30'$. The base is 14.6 meters long.
    a. Find the length of a leg of the triangle.
    b. Find the altitude of the triangle.
    c. What is the area of the triangle?

22. **Geometry** A regular hexagon is inscribed in a circle with diameter 6.4 centimeters.

a. What is the apothem of the hexagon?

b. Find the length of a side of the hexagon.

c. Find the perimeter of the hexagon.

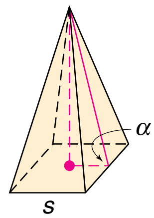
d. The area of a regular polygon equals one half times the perimeter of the polygon times the apothem. Find the area of the polygon.

Applications and Problem Solving

Real World Application

23. **Engineering** The escalator at St. Petersburg Metro in Russia has a vertical rise of 195.8 feet. If the angle of elevation of the escalator is 10° 21′ 36″, find the length of the escalator.

24. **Critical Thinking** Write a formula for the volume of the regular pyramid at the right in terms of α and s the length of each side of the base.



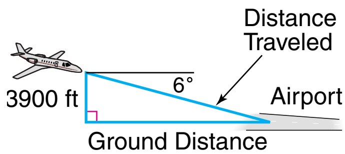
25. **Fire Fighting** The longest truck-mounted ladder used by the Dallas Fire Department is 108 feet long and consists of four hydraulic sections. Gerald Travis, aerial expert for the department, indicates that the optimum operating angle of this ladder is 60°. The fire fighters find they need to reach the roof of an 84-foot burning building. Assume the ladder is mounted 8 feet above the ground.

a. Draw a labeled diagram of the situation.

b. How far from the building should the base of the ladder be placed to achieve the optimum operating angle?

c. How far should the ladder be extended to reach the roof?

26. **Aviation** When a 757 passenger jet begins its descent to the Ronald Reagan International Airport in Washington, D.C., it is 3900 feet from the ground. Its angle of descent is 6°.

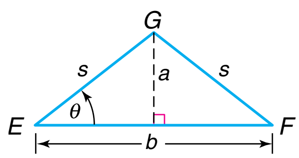


a. What is the plane's ground distance to the airport?

b. How far must the plane fly to reach the runway?

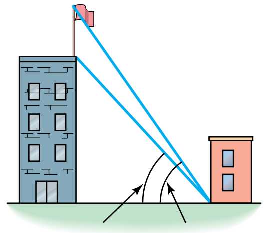
27. **Boat Safety** The Cape Hatteras lighthouse on the North Carolina coast was built in 1870 and rises 208 feet above sea level. From the top of the lighthouse, the lighthouse keeper observes a yacht and a barge along the same line of sight. The angle of depression for the yacht is 20°, and the angle of depression for the barge is 12° 30′. For safety purposes, the keeper thinks that the two sea vessels should be at least 300 feet apart. If they are less than 300 feet, she plans to sound the horn. How far apart are these vessels? Does the keeper have to sound the horn?

28. **Critical Thinking** Derive two formulas for the length of the altitude a of the triangle shown at the right, given that b, s, and θ are known. Justify each of the steps you take in your reasoning.



29. Recreation Latasha and Markisha are flying kites on a windy spring day. Latasha has released 250 feet of string, and Markisha has released 225 feet of string. The angle that Latasha's kite string makes with the horizontal is 35°. The angle that Markisha's kite string makes with the horizontal is 42°. Which kite is higher and by how much?

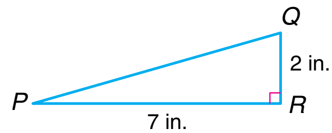
30. Architecture A flagpole 40 feet high stands on top of the Wentworth Building. From a point in front of Bailey's Drugstore, the angle of elevation for the top of the pole is 54° 54′, and the angle of elevation for the bottom of the pole is 47° 30′. How high is the building?



Mixed Review

31. Find the values of the six trigonometric functions for a 120° angle using the unit circle. *(Lesson 5-3)*

32. Find the sine, cosine, and tangent ratios for $\angle P$. *(Lesson 5-2)*



33. Write 43° 15′ 35″ as a decimal to the nearest thousandth. *(Lesson 5-1)*

34. Graph $y \leq |x + 2|$. *(Lesson 3-3)*

35. Consumerism Kareem and Erin went shopping for school supplies. Kareem bought 3 notebooks and 2 packages of pencils for \$5.80. Erin bought 4 notebooks and 1 package of pencils for \$6.20. What is the cost of one notebook? What is the cost of one package of pencils? *(Lesson 2-1)*

36. SAT/ACT Practice An automobile travels m miles in h hours. At this rate, how far will it travel in x hours?

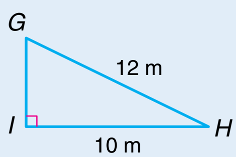
A $\frac{m}{x}$ **B** $\frac{m}{xh}$ **C** $\frac{m}{h}$ **D** $\frac{mh}{x}$ **E** $\frac{mx}{h}$

MID-CHAPTER QUIZ

1. Change 34.605° to degrees, minutes, and seconds. (Lesson 5-1)

2. If a −400° angle is in standard position, determine a coterminal angle that is between 0° and 360°. State the quadrant in which the terminal side lies. (Lesson 5-1)

3. Find the six trigonometric functions for $\angle G$. (Lesson 5-2)



4. Find the values of the six trigonometric functions for angle θ in standard position if a point with coordinates (2, −5) lies on its terminal side. (Lesson 5-3)

5. National Landmarks Suppose the angle of elevation of the sun is 27.8°. Find the length of the shadow made by the Washington Monument, which is 550 feet tall. (Lesson 5-4)

Extra Practice See p. A34.

5-5

Solving Right Triangles

OBJECTIVES

- Evaluate inverse trigonometric functions.
- Find missing angle measurements.
- Solve right triangles.

Real World Application

SECURITY A security light is being installed outside a loading dock. The light is mounted 20 feet above the ground. The light must be placed at an angle so that it will illuminate the end of the parking lot. If the end of the parking lot is 100 feet from the loading dock, what should be the angle of depression of the light? *This problem will be solved in Example 4.*

In Lesson 5-3, you learned to use the unit circle to determine the value of trigonometric functions. Some of the frequently-used values are listed below.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

θ	210°	225°	240°	270°	300°	315°	330°	360°
sin θ	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos θ	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan θ	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

Sometimes you know a trigonometric value of an angle, but not the angle. In this case, you need to use an **inverse** of the trigonometric function. The inverse of the sine function is the **arcsine relation.**

An equation such as $\sin x = \frac{\sqrt{3}}{2}$ can be written as $x = \arcsin \frac{\sqrt{3}}{2}$, which is read "$x$ is an angle whose sine is $\frac{\sqrt{3}}{2}$," or "x equals the arcsine of $\frac{\sqrt{3}}{2}$." The solution, x, consists of all angles that have $\frac{\sqrt{3}}{2}$ as the value of sine x.

Similarly, the inverse of the cosine function is the **arccosine relation,** and the inverse of the tangent function is the **arctangent relation.**

The equations in each row of the table below are equivalent. You can use these equations to rewrite trigonometric expressions.

Inverses of the Trigonometric Functions

Trigonometric Function	Inverse Trigonometric Relation
$y = \sin x$	$x = \sin^{-1} y$ or $x = \arcsin y$
$y = \cos x$	$x = \cos^{-1} y$ or $x = \arccos y$
$y = \tan x$	$x = \tan^{-1} y$ or $x = \arctan y$

Examples

1 Solve each equation.

a. $\sin x = \frac{\sqrt{3}}{2}$

If $\sin x = \frac{\sqrt{3}}{2}$, then x is an angle whose sine is $\frac{\sqrt{3}}{2}$.

$x = \arcsin \frac{\sqrt{3}}{2}$

From the table on page 305, you can determine that x equals 60°, 120°, or any angle coterminal with these angles.

b. $\cos x = -\frac{\sqrt{2}}{2}$

If $\cos x = -\frac{\sqrt{2}}{2}$, then x is an angle whose cosine is $-\frac{\sqrt{2}}{2}$.

$x = \arccos -\frac{\sqrt{2}}{2}$

From the table, you can determine that x equals 135°, 225°, or any angle coterminal with these angles.

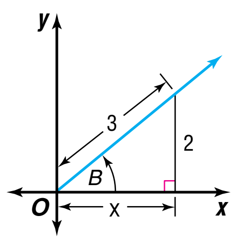
2 Evaluate each expression. Assume that all angles are in Quadrant I.

a. $\tan\left(\tan^{-1} \frac{6}{11}\right)$

Let $A = \tan^{-1} \frac{6}{11}$. Then $\tan A = \frac{6}{11}$ by the definition of inverse. Therefore, by substitution, $\tan\left(\tan^{-1} \frac{6}{11}\right) = \frac{6}{11}$.

b. $\cos\left(\arcsin \frac{2}{3}\right)$

Let $B = \arcsin \frac{2}{3}$. Then $\sin B = \frac{2}{3}$ by the definition of inverse. Draw a diagram of the $\angle B$ in Quadrant I.



$r^2 = x^2 + y^2$ *Pythagorean Theorem*

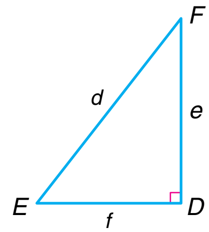
$3^2 = x^2 + 2^2$ *Substitute 3 for r and 2 for y.*

$\sqrt{5} = x$ *Take the square root of each side. Disregard the negative root.*

Since $\cos = \frac{\text{side adjacent}}{\text{hypotenuse}}$, $\cos B = \frac{\sqrt{5}}{3}$ and $\cos\left(\arcsin \frac{2}{3}\right) = \frac{\sqrt{5}}{3}$.

Inverse trigonometric relations can be used to find the measure of angles of right triangles. Calculators can be used to find values of the inverse trigonometric relations.

Example 3 **If $f = 17$ and $d = 32$, find E.**



In this problem, you want to know the measure of an acute angle in a right triangle. You know the side adjacent to the angle and the hypotenuse. The cosine function relates the side adjacent to the angle and the hypotenuse.

Remember that in trigonometry the measure of an angle is symbolized by the angle vertex letter.

$\cos E = \frac{f}{d}$ — *$\cos = \frac{\text{side adjacent}}{\text{hypotenuse}}$*

$\cos E = \frac{17}{32}$ — *Substitute 17 for f and 32 for d.*

$E = \cos^{-1}\frac{17}{32}$ — *Definition of inverse*

$E \approx 57.91004874$ — *Use a calculator.*

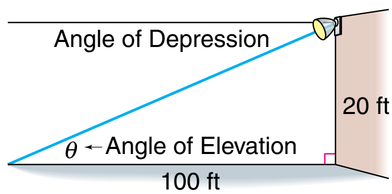
Therefore, E measures about 57.9°.

Trigonometry can be used to find the angle of elevation or the angle of depression.

Example 4 Real World Application

SECURITY **Refer to the application at the beginning of the lesson. What should be the angle of depression of the light?**

The angle of depression from the light and the angle of elevation to the light are equal in measure. To find the angle of elevation, use the tangent function.



$\tan \theta = \frac{20}{100}$ — *$\tan = \frac{\text{side opposite}}{\text{side adjacent}}$*

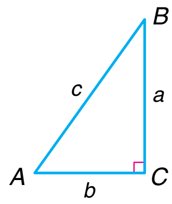
$\theta = \tan^{-1}\frac{20}{100}$ — *Definition of inverse*

$\theta \approx 11.30993247$ — *Use a calculator.*

The angle of depression should be about 11.3°.

You can use trigonometric functions and inverse relations to solve right triangles. To **solve a triangle** means to find all of the measures of its sides and angles. Usually, two measures are given. Then you can find the remaining measures.

Example 5 **Solve each triangle described, given the triangle at the right.**



a. $A = 33°$, $b = 5.8$

Find B.

$33° + B = 90°$ *Angles A and B are complementary.*

$B = 57°$

Whenever possible, use measures given in the problem to find the unknown measures.

Find a.

$$\tan A = \frac{a}{b}$$
$$\tan 33° = \frac{a}{5.8}$$
$$5.8 \tan 33° = a$$
$$3.766564041 \approx a$$

Find c.

$$\cos A = \frac{b}{c}$$
$$\cos 33° = \frac{5.8}{c}$$
$$c \cos 33° = 5.8$$
$$c = \frac{5.8}{\cos 33°}$$
$$c \approx 6.915707098$$

Therefore, $B = 57°$, $a \approx 3.8$, and $c \approx 6.9$.

b. $a = 23$, $c = 45$

Find b.

$$a^2 + b^2 = c^2$$
$$23^2 + b^2 = 45^2$$
$$b = \sqrt{1496}$$
$$b \approx 38.67815921$$

Find A.

$$\sin A = \frac{a}{c}$$
$$\sin A = \frac{23}{45}$$
$$A = \sin^{-1}\frac{23}{45}$$
$$A \approx 30.73786867$$

Find B.

$$30.73786867 + B \approx 90$$
$$B \approx 59.26213133$$

Therefore, $b \approx 38.7$, $A \approx 30.7°$, and $B \approx 59.3°$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Tell** whether the solution to each equation is an angle measure or a linear measurement.
   a. $\tan 34° 15' = \frac{x}{12}$ b. $\tan x = 3.284$
2. **Describe** the relationship of the two acute angles of a right triangle.
3. **Counterexample** You can usually solve a right triangle if you know two measures besides the right angle. Draw a right triangle and label two measures other than the right angle such that you cannot solve the triangle.
4. **You Decide** Marta and Rebecca want to determine the degree measure of angle ϑ if $\cos \vartheta = 0.9876$. Marta tells Rebecca to press [2nd] [COS^{-1}] .9876 on the calculator. Rebecca disagrees. She says to press [COS] .9876 [)] [x^{-1}]. Who is correct? Explain.

Guided Practice

Solve each equation if $0° \leq x \leq 360°$.

5. $\cos x = \frac{1}{2}$

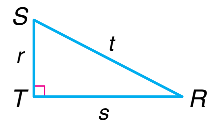
6. $\tan x = \frac{-\sqrt{3}}{3}$

Evaluate each expression. Assume that all angles are in Quadrant I.

7. $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

8. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

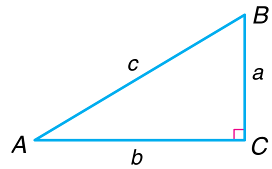
Solve each problem. Round to the nearest tenth.



9. If $r = 7$ and $s = 10$, find R.

10. If $r = 12$ and $t = 20$, find S.

Solve each triangle described, given the triangle at the right. Round to the nearest tenth if necessary.

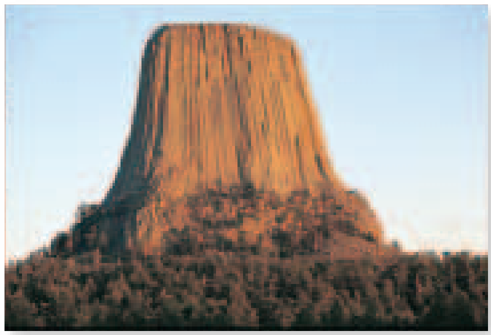


11. $B = 78°$, $a = 41$

12. $a = 11$, $b = 21$

13. $A = 32°$, $c = 13$

14. **National Monuments** In 1906, Teddy Roosevelt designated Devils Tower National Monument in northeast Wyoming as the first national monument in the United States. The tower rises 1280 feet above the valley of the Bell Fourche River.



Devils Tower National Monument

a. If the shadow of the tower is 2100 feet long at a certain time, find the angle of elevation of the sun.

b. How long is the shadow when the angle of elevation of the sun is 38°?

c. If a person at the top of Devils Tower sees a hiker at an angle of depression of 65°, how far is the hiker from the base of Devils Tower?

EXERCISES

Practice

Solve each equation if $0° \leq x \leq 360°$.

15. $\sin x = 1$

16. $\tan x = -\sqrt{3}$

17. $\cos x = \frac{\sqrt{3}}{2}$

18. $\cos x = 0$

19. $\sin x = -\frac{\sqrt{2}}{2}$

20. $\tan x = -1$

21. Name four angles whose sine equals $\frac{1}{2}$.

Evaluate each expression. Assume that all angles are in Quadrant I.

22. $\cos\left(\arccos\frac{4}{5}\right)$

23. $\tan\left(\tan^{-1}\frac{2}{3}\right)$

24. $\sec\left(\cos^{-1}\frac{2}{5}\right)$

25. $\csc(\arcsin 1)$

26. $\tan\left(\cos^{-1}\frac{5}{13}\right)$

27. $\cos\left(\sin^{-1}\frac{2}{5}\right)$

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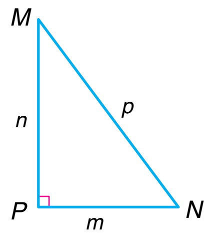
Solve each problem. Round to the nearest tenth.



28. If $n = 15$ and $m = 9$, find N.

29. If $m = 8$ and $p = 14$, find M.

30. If $n = 22$ and $p = 30$, find M.

31. If $m = 14.3$ and $n = 18.8$, find N.

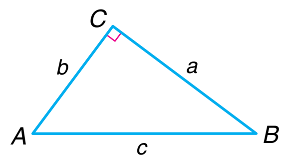
32. If $p = 17.1$ and $m = 7.2$, find N.

33. If $m = 32.5$ and $p = 54.7$, find M.

34. **Geometry** If the legs of a right triangle are 24 centimeters and 18 centimeters long, find the measures of the acute angles.

35. **Geometry** The base of an isosceles triangle is 14 inches long. Its height is 8 inches. Find the measure of each angle of the triangle.

Solve each triangle described, given the triangle at the right. Round to the nearest tenth, if necessary.



36. $a = 21$, $c = 30$

37. $A = 35°$, $b = 8$

38. $B = 47°$, $b = 12.5$

39. $a = 3.8$, $b = 4.2$

40. $c = 9.5$, $b = 3.7$

41. $a = 13.3$, $A = 51.5°$

42. $B = 33°$, $c = 15.2$

43. $c = 9.8$, $A = 14°$

Applications and Problem Solving

Real World Application

44. **Railways** The steepest railway in the world is the Katoomba Scenic Railway in Australia. The passenger car is pulled up the mountain by twin steel cables. It travels along the track 1020 feet to obtain a change in altitude of 647 feet.

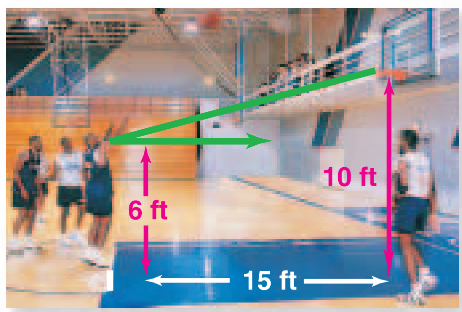
a. Find the angle of elevation of the railway.

b. How far does the car travel in a horizontal direction?

45. **Critical Thinking** Explain why each expression is impossible.

a. $\sin^{-1} 2.4567$ b. $\sec^{-1} 0.5239$ c. $\cos^{-1} (-3.4728)$

46. **Basketball** The rim of a basketball hoop is 10 feet above the ground. The free-throw line is 15 feet from the basket rim. If the eyes of a basketball player are 6 feet above the ground, what is the angle of elevation of the player's line of sight when shooting a free throw to the rim of the basket?



47. **Road Safety** Several years ago, a section on I-75 near Cincinnati, Ohio, had a rise of 8 meters per 100 meters of horizontal distance. However, there were numerous accidents involving large trucks on this section of highway. Civil engineers decided to reconstruct the highway so that there is only a rise of 5 meters per 100 meters of horizontal distance.

a. Find the original angle of elevation.

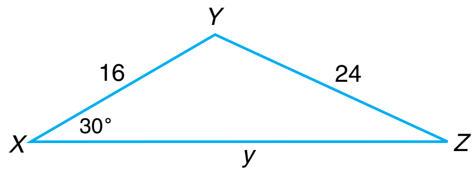
b. Find the new angle of elevation.

48. Air Travel At a local airport, a light that produces a powerful white-green beam is placed on the top of a 45-foot tower. If the tower is at one end of the runway, find the angle of depression needed so that the light extends to the end of the 2200-foot runway.

49. Civil Engineering Highway curves are usually banked or tilted inward so that cars can negotiate the curve more safely. The proper banking angle θ for a car making a turn of radius r feet at a velocity of v feet per second is given by the equation is $\tan \theta = \frac{v^2}{gr}$. In this equation, g is the acceleration due to gravity or 32 feet per second squared. An engineer is designing a curve with a radius of 1200 feet. If the speed limit on the curve will be 65 miles per hour, at what angle should the curve be banked? (*Hint*: Change 65 miles per hour to feet per second.)

50. Physics According to Snell's Law, $\frac{\sin \theta_i}{\sin \theta_r} = n$, where θ_i is the angle of incidence, θ_r is the angle of refraction, and n is the index of refraction. The index of refraction for a diamond is 2.42. If a beam of light strikes a diamond at an angle of incidence of 60°, find the angle of refraction.

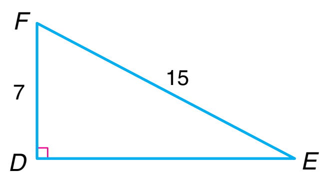
51. Critical Thinking Solve the triangle. (*Hint*: Draw the altitude from Y.)



Mixed Review

52. Aviation A traffic helicopter is flying 1000 feet above the downtown area. To the right, the pilot sees the baseball stadium at an angle of depression of 63°. To the left, the pilot sees the football stadium at an angle of depression of 18°. Find the distance between the two stadiums. *(Lesson 5-4)*

53. Find the six trigonometric ratios for $\angle F$. *(Lesson 5-2)*



54. Approximate the real zeros of the function $f(x) = 3x^3 - 16x^2 + 12x + 6$ to the nearest tenth. *(Lesson 4-5)*

55. Determine whether the graph of $y^3 - x^2 = 2$ is symmetric with respect to the x-axis, y-axis, the graph of $y = x$, the graph of $y = -x$, or none of these. *(Lesson 3-1)*

56. Use a reflection matrix to find the coordinates of the vertices of a pentagon reflected over the y-axis if the coordinates of the vertices of the pentagon are $(-5, -3)$, $(-5, 4)$, $(-3, 6)$, $(-1, 3)$, and $(-2, -2)$. *(Lesson 2-4)*

57. Find the sum of the matrices $\begin{bmatrix} 4 & -3 & 2 \\ 8 & -2 & 0 \\ 9 & 6 & -3 \end{bmatrix}$ and $\begin{bmatrix} -2 & 2 & -2 \\ -5 & 1 & 1 \\ -7 & 2 & -2 \end{bmatrix}$. *(Lesson 2-3)*

Extra Practice See p. A35.

CONTENTS

58. Write a linear equation of best fit for a set of data using the ordered pairs (1880, 42.5) and (1950, 22.2). *(Lesson 1-6)*

59. Write the equation $2x + 5y - 10 = 0$ in slope-intercept form. Then name the slope and y-intercept. *(Lesson 1-5)*

60. **SAT/ACT Practice** The Natural Snack Company mixes a pounds of peanuts that cost b cents per pound with c pounds of rice crackers that cost d cents per pound to make Oriental Peanut Mix. What should the price in cents for a pound of Oriental Peanut Mix be if the company wants to make a profit of 10¢ per pound?

A $\frac{ab + cd}{a + c} + 10$ **B** $\frac{b + d}{a + c} + 10$ **C** $\frac{ab + cd}{a + c} + 0.10$

D $\frac{b + d}{a + c} + 0.10$ **E** $\frac{b + d + 10}{a + c}$

CAREER CHOICES

Architecture

Are you creative and concerned with accuracy and detail? Do you like to draw and design new things? You may want to consider a career in architecture.

An architect plans, designs, and oversees the construction of all types of buildings, a job that is very complex. An architect needs to stay current with new construction methods and design. An architect must also be knowledgeable about engineering principles.

As an architect, you would work closely with others such as consulting engineers and building contractors. Your projects could include large structures such as shopping malls or small structures such as single-family houses. There are also specialty fields in architecture such as interior design, landscape architecture, and products and material design.

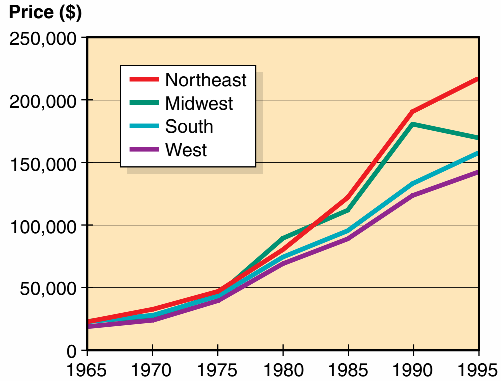
Career Overview

Degree Preferred:
bachelor's degree in architecture

Related Courses:
mathematics, physics, art, computer science

Outlook:
number of jobs expected to increase as fast as the average through the year 2006

Average Sale Prices of New Single-Family Houses by Region



Source: *The New York Times Almanac*

internet CONNECTION For more information on careers in architecture, visit: **www.amc.glencoe.com**

5-6

The Law of Sines

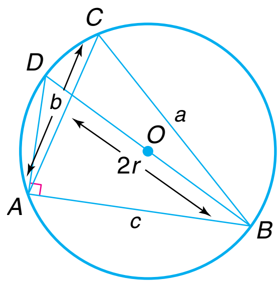
OBJECTIVES

- Solve triangles by using the Law of Sines if the measures of two angles and a side are given.
- Find the area of a triangle if the measures of two sides and the included angle or the measures of two angles and a side are given.

Real World Application

BASEBALL A baseball fan is sitting directly behind home plate in the last row of the upper deck of Comiskey Park in Chicago. The angle of depression to home plate is 29° 54′, and the angle of depression to the pitcher's mound is 24° 12′. In major league baseball, the distance between home plate and the pitcher's mound is 60.5 feet. How far is the fan from home plate? *This problem will be solved in Example 2.*

The **Law of Sines** can be used to solve triangles that are not right triangles. Consider $\triangle ABC$ inscribed in circle O with diameter $\overline{DB}$. Let $2r$ be the measure of the diameter. Draw $\overline{AD}$. Then $\angle D \cong \angle C$ since they intercept the same arc. So, $\sin D = \sin C$. $\angle DAB$ is inscribed in a semicircle, so it is a right angle. $\sin D = \frac{c}{2r}$. Thus, since $\sin D = \sin C$, it follows that $\sin C = \frac{c}{2r}$ or $\frac{c}{\sin C} = 2r$.



Similarly, by drawing diameters through A and C, $\frac{b}{\sin B} = 2r$ and $\frac{a}{\sin A} = 2r$. Since each rational expression equals 2r, the following is true.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

These equations state that the ratio of the length of any side of a triangle to the sine of the angle opposite that side is a constant for a given triangle. These equations are collectively called the Law of Sines.

Law of Sines

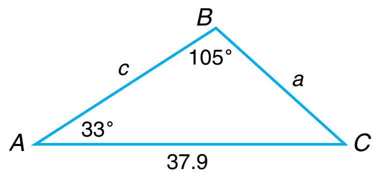
Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of the sides opposite the angles with measures A, B, and C, respectively. Then, the following is true.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

From geometry, you know that a unique triangle can be formed if you know the measures of two angles and the included side (ASA) or the measures of two angles and the non-included side (AAS). Therefore, there is one unique solution when you use the Law of Sines to solve a triangle given the measures of two angles and one side. *In Lesson 5-7, you will learn to use the Law of Sines when the measures of two sides and a nonincluded angle are given.*

Examples

1 **Solve $\triangle ABC$ if $A = 33°$, $B = 105°$, and $b = 37.9$.**



First, find the measure of $\angle C$.
$C = 180° - (33° + 105°)$ or $42°$

Use the Law of Sines to find a and c.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 33°} = \frac{37.9}{\sin 105°} \qquad \frac{c}{\sin 42°} = \frac{37.9}{\sin 105°}$$

$$a = \frac{37.9 \sin 33°}{\sin 105°} \qquad c = \frac{37.9 \sin 42°}{\sin 105°}$$

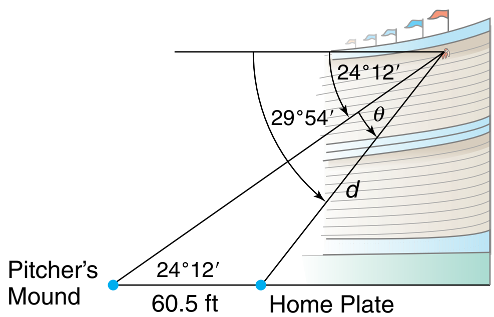
$a \approx 21.36998397$ *Use a calculator.* $c \approx 26.25465568$ *Use a calculator.*

Therefore, $C = 42°$, $a \approx 21.4$, and $c \approx 26.3$.

2 **BASEBALL** **Refer to the application at the beginning of the lesson. How far is the fan from home plate?**

Real World Application

Make a diagram for the problem. Remember that the angle of elevation is congruent to the angle of depression, because they are alternate interior angles.



First, find θ.

$\theta = 29° \, 54' - 24° \, 12'$ or $5° \, 42'$

Use the Law of Sines to find d.

$$\frac{d}{\sin 24° \, 12'} = \frac{60.5}{\sin 5° \, 42'}$$

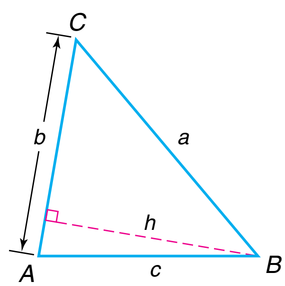
$$d = \frac{60.5 \sin 24° \, 12'}{\sin 5° \, 42'}$$

$d \approx 249.7020342$ *Use a calculator.*

The fan is about 249.7 feet from home plate.

Use K for area instead of A to avoid confusion with angle A.

The area of any triangle can be expressed in terms of two sides of a triangle and the measure of the included angle. Suppose you know the measures of $\overline{AC}$ and $\overline{AB}$ and the measure of the included $\angle A$ in $\triangle ABC$. Let K represent the measure of the area of $\triangle ABC$, and let h represent the measure of the altitude from B. Then $K = \frac{1}{2}bh$. But, $\sin A = \frac{h}{c}$ or $h = c \sin A$. If you substitute $c \sin A$ for h, the result is the following formula.



$$K = \frac{1}{2}bc \sin A$$

If you drew altitudes from A and C, you could also develop two similar formulas.

Area of Triangles

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of the sides opposite the angles with measurements A, B, and C, respectively. Then the area K can be determined using one of the following formulas.

$$K = \frac{1}{2}bc \sin A \qquad K = \frac{1}{2}ac \sin B$$
$$K = \frac{1}{2}ab \sin C$$

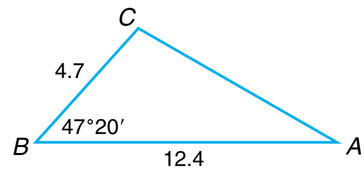
Example 3 **Find the area of $\triangle ABC$ if $a = 4.7$, $c = 12.4$, and $B = 47° 20'$.**

$K = \frac{1}{2}ac \sin B$

$K = \frac{1}{2}(4.7)(12.4) \sin 47° 20'$

$K \approx 21.42690449$ *Use a calculator.*

The area of $\triangle ABC$ is about 21.4 square units.



You can also find the area of a triangle if you know the measures of one side and two angles of the triangle. By the Law of Sines, $\frac{b}{\sin B} = \frac{c}{\sin C}$ or $b = \frac{c \sin B}{\sin C}$. If you substitute $\frac{c \sin B}{\sin C}$ for b in $K = \frac{1}{2}bc \sin A$, the result is $K = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}$. Two similar formulas can be developed.

Area of Triangles

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of the sides opposite the angles with measurements A, B, and C respectively. Then the area K can be determined using one of the following formulas.

$$K = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} \qquad K = \frac{1}{2}b^2 \frac{\sin A \sin C}{\sin B}$$
$$K = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}$$

Example 4 **Find the area of $\triangle DEF$ if $d = 13.9$, $D = 34.4°$, and $E = 14.8°$.**

First find the measure of $\angle F$.
$F = 180° - (34.4° + 14.8°)$ or $130.8°$

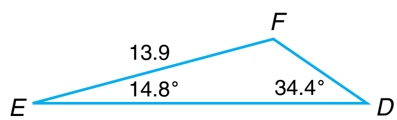
Then, find the area of the triangle.

$K = \frac{1}{2}d^2 \frac{\sin E \sin F}{\sin D}$

$K = \frac{1}{2}(13.9)^2 \frac{\sin 14.8° \sin 130.8°}{\sin 34.4°}$

$K \approx 33.06497958$ *Use a calculator.*

The area of $\triangle DEF$ is about 33.1 square units.

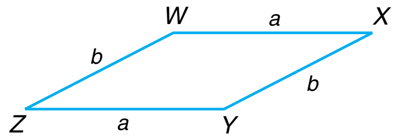


Check for Understanding

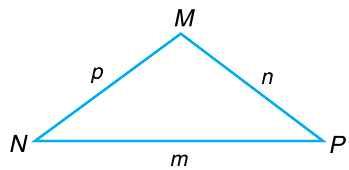
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Show** that the Law of Sines is true for a 30°-60° right triangle.
2. **Draw and label** a triangle that has a unique solution and can be solved using the Law of Sines.
3. **Write** a formula for the area of parallelogram *WXYZ* in terms of *a, b,* and *X.*



4. **You Decide** Roderick says that triangle *MNP* has a unique solution if *M, N,* and *m* are known. Jane disagrees. She says that a triangle has a unique solution if *M, N,* and *p* are known. Who is correct? Explain.



Guided Practice

Solve each triangle. Round to the nearest tenth.

5. $A = 40°$, $B = 59°$, $c = 14$
6. $a = 8.6$, $A = 27.3°$, $B = 55.9°$
7. If $B = 17°\ 55'$, $C = 98°\ 15'$, and $a = 17$, find c.

Find the area of each triangle. Round to the nearest tenth.

8. $A = 78°$, $b = 14$, $c = 12$
9. $A = 22°$, $B = 105°$, $b = 14$
10. **Baseball** Refer to the application at the beginning of the lesson. How far is the baseball fan from the pitcher's mound?

Exercises

Practice

Solve each triangle. Round to the nearest tenth.

11. $A = 40°$, $C = 70°$, $a = 20$
12. $B = 100°$, $C = 50°$, $c = 30$
13. $b = 12$, $A = 25°$, $B = 35°$
14. $A = 65°$, $B = 50°$, $c = 12$
15. $a = 8.2$, $B = 24.8°$, $C = 61.3°$
16. $c = 19.3$, $A = 39°\ 15'$, $C = 64°\ 45'$
17. If $A = 37°\ 20'$, $B = 51°\ 30'$, and $c = 125$, find b.
18. What is a if $b = 11$, $B = 29°\ 34'$, and $C = 23°\ 48'$?

Find the area of each triangle. Round to the nearest tenth.

19. $A = 28°$, $b = 14$, $c = 9$
20. $a = 5$, $B = 37°$, $C = 84°$
21. $A = 15°$, $B = 113°$, $b = 7$
22. $b = 146.2$, $c = 209.3$, $A = 62.2°$
23. $B = 42.8°$, $a = 12.7$, $c = 5.8$
24. $a = 19.2$, $A = 53.8°$, $C = 65.4°$
25. **Geometry** The adjacent sides of a parallelogram measure 14 centimeters and 20 centimeters, and one angle measures 57°. Find the area of the parallelogram.
26. **Geometry** A regular pentagon is inscribed in a circle whose radius measures 9 inches. Find the area of the pentagon.
27. **Geometry** A regular octagon is inscribed in a circle with radius of 5 feet. Find the area of the octagon.

www.amc.glencoe.com/self_check_quiz

Applications and Problem Solving

Real World Application

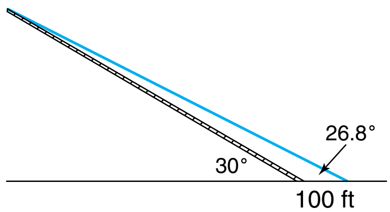
28. **Landscaping** A landscaper wants to plant begonias along the edges of a triangular plot of land in Winton Woods Park. Two of the angles of the triangle measure 95° and 40°. The side between these two angles is 80 feet long.
   a. Find the measure of the third angle.
   b. Find the length of the other two sides of the triangle.
   c. What is the perimeter of this triangular plot of land?

29. **Critical Thinking** For $\triangle MNP$ and $\triangle RST$, $\angle M \cong \angle R$, $\angle N \cong \angle S$, and $\angle P \cong \angle T$. Use the Law of Sines to show $\triangle MNP \sim \triangle RST$.

30. **Architecture** The center of the Pentagon in Arlington, Virginia, is a courtyard in the shape of a regular pentagon. The pentagon could be inscribed in a circle with radius of 300 feet. Find the area of the courtyard.

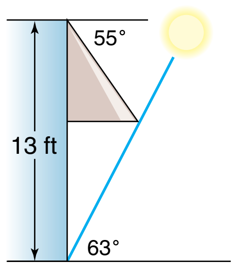
31. **Ballooning** A hot air balloon is flying above Groveburg. To the left side of the balloon, the balloonist measures the angle of depression to the Groveburg soccer fields to be 20° 15′. To the right side of the balloon, the balloonist measures the angle of depression to the high school football field to be 62° 30′. The distance between the two athletic complexes is 4 miles.
   a. Find the distance from the balloon to the soccer fields.
   b. What is the distance from the balloon to the football field?

32. **Cable Cars** The Duquesne Incline is a cable car in Pittsburgh, Pennsylvania, which transports passengers up and down a mountain. The track used by the cable car has an angle of elevation of 30°. The angle of elevation to the top of the track from a point that is horizontally 100 feet from the base of the track is about 26.8°. Find the length of the track.



33. **Air Travel** In order to avoid a storm, a pilot starts the flight 13° off course. After flying 80 miles in this direction, the pilot turns the plane to head toward the destination. The angle formed by the course of the plane during the first part of the flight and the course during the second part of the flight is 160°.
   a. What is the distance of the flight?
   b. Find the distance of a direct flight to the destination.

34. **Architecture** An architect is designing an overhang above a sliding glass door. During the heat of the summer, the architect wants the overhang to prevent the rays of the sun from striking the glass at noon. The overhang has an angle of depression of 55° and starts 13 feet above the ground. If the angle of elevation of the sun during this time is 63°, how long should the architect make the overhang?



CONTENTS

35. Critical Thinking Use the Law of Sines to show that each statement is true for any $\triangle ABC$.

a. $\frac{a}{b} = \frac{\sin A}{\sin B}$

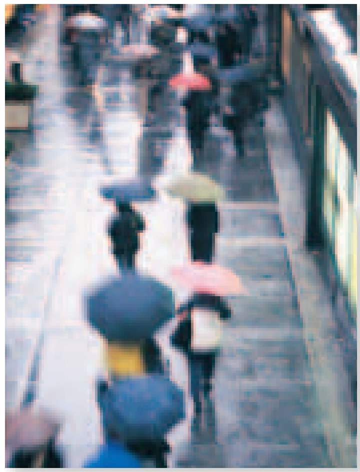
b. $\frac{a - c}{c} = \frac{\sin A - \sin C}{\sin C}$

c. $\frac{a + c}{a - c} = \frac{\sin A + \sin C}{\sin A - \sin C}$

d. $\frac{b}{a + b} = \frac{\sin B}{\sin A + \sin B}$

Mixed Review

36. Meteorology If raindrops are falling toward Earth at a speed of 45 miles per hour and a horizontal wind is blowing at a speed of 20 miles per hour, at what angle do the drops hit the ground? *(Lesson 5-5)*



37. Suppose θ is an angle in standard position whose terminal side lies in Quadrant IV. If $\sin \theta = -\frac{1}{6}$, find the values of the remaining five trigonometric functions for θ. *(Lesson 5-3)*

38. Identify all angles that are coterminal with an 83° angle. *(Lesson 5-1)*

39. Business A company is planning to buy new carts to store merchandise. The owner believes they need at least 2 standard carts and at least 4 deluxe carts. The company can afford to purchase a maximum of 15 carts at this time; however, the supplier has only 8 standard carts and 11 deluxe carts in stock. Standard carts can hold up to 100 pounds of merchandise, and deluxe carts can hold up to 250 pounds of merchandise. How many standard carts and deluxe carts should be purchased to maximize the amount of merchandise that can be stored? *(Lesson 2-7)*

40. Solve the system of equations algebraically. *(Lesson 2-2)*

$4x + y + 2z = 0$
$3x + 4y - 2z = 20$
$-2x + 5y + 3z = 14$

41. Graph $-6 \leq 3x - y \leq 12$. *(Lesson 1-8)*

42. SAT Practice Eight cubes, each with an edge of length one inch, are positioned together to create a large cube. What is the difference in the surface area of the large cube and the sum of the surface areas of the small cubes?

A 24 in^2
B 16 in^2
C 12 in^2
D 8 in^2
E 0 in^2

Extra Practice See p. A35.

HISTORY of MATHEMATICS

ANGLES

When someone uses the word "angle", what images does that conjure up in your mind? An angle seems like a simple figure, but historically mathematicians, and even philosophers, have engaged in trying to describe or define an angle. This textbook says, "an angle may be generated by the rotation of two rays that share a fixed endpoint known as the vertex." Let's look at various ideas about angles throughout history.

Autumn Borts

Early Evidence Babylonians (4000–3000 B.C.) were some of the first peoples to leave samples of their use of geometry in the form of inscribed clay tablets.

The first written mathematical work containing definitions for angles was **Euclid's** *The Elements.* Little is known about the life of **Euclid** (about 300 B.C.), but his thirteen-volume work, *The Elements,* has strongly influenced the teaching of geometry for over 2000 years. The first copy of *The Elements* was printed by modern methods in 1482 and has since been edited and translated into over 1000 editions. In *Book I* of *The Elements,* Euclid presents the definitions for various types of angles.

Euclid's definition of a plane angle differed from an earlier Greek idea that an angle was a deflection or a breaking of lines.

Greek mathematicians were not the only scholars interested in angles. **Aristotle** (384–322 B.C.) had devised three categories in which to place mathematical concepts—a quantity, a quality, or a relation. Greek philosophers argued as to which category an angle belonged. **Proclus** (410–485) felt that an angle was a combination of the three, saying "it needs the quantity involved in magnitude, thereby becoming susceptible of equality, inequality, and the like; it needs the quality given it by its form; and lastly, the relation subsisting between the lines or the planes bounding it."

The Renaissance In 1634, Pierre Herigone first used "$<$" as a symbol for an angle in his book *Cursus Mathematicus.* This symbol was already being used for "less than," so, in 1657, **William Oughtred** used the symbol "$\angle$" in his book *Trigonometria.*

Modern Era Various symbols for angle, including $\wedge$, ⦣, $\widehat{ab}$, and $\widehat{ABC}$, were used during the 1700s and 1800s. In 1923, the National Committee on Mathematical Requirements recommended that "$\angle$" be used as a standard symbol in the U.S.

Today, artists like **Autumn Borts** use angles in their creation of Native American pottery. Ms. Borts adorns water jars with carved motifs of both traditional and contemporary designs. She is carrying on the Santa Clara style of pottery and has been influenced by her mother, grandmother, and great grandmother.

ACTIVITIES

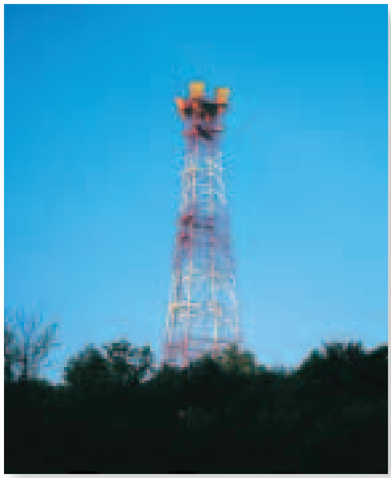
1. In a previous course, you have probably drawn triangles in a plane and measured the interior angles to find the angle sum of the triangles. Triangles can also be constructed on a sphere. Get a globe. Use tape and string to form at least three different triangles. Measure the interior angles of the triangles. What appears to be true about the sum of the angles?
2. Research Euclid's famous work, *The Elements.* Find and list any postulates he wrote about angles.
3. interNET CONNECTION Find out more about the personalities referenced in this article and others who contributed to the history of angles. Visit **www.amc.glencoe.com**

5-7 The Ambiguous Case for the Law of Sines

OBJECTIVES

- Determine whether a triangle has zero, one, or two solutions.
- Solve triangles using the Law of Sines.

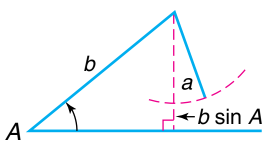
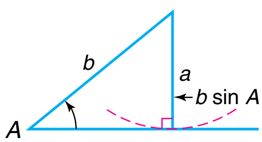
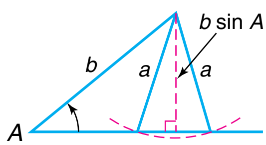
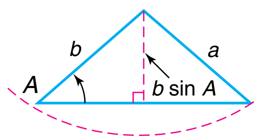
TOURISM Visitors near a certain national park can tune to a local radio station to find out about the activities that are happening in the park. The transmission tower for the radio station is along Park Road about 30 miles from the intersection of this road and the interstate. The interstate and the road form a 47° angle. If the transmitter has a range of 25 miles, how far along the interstate can the passengers in a car hear the broadcast? *This problem will be solved in Example 3.*

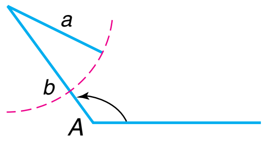
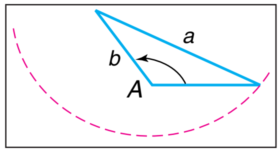


From geometry, you know that the measures of two sides and a nonincluded angle do not necessarily define a unique triangle. However, one of the following will be true.

1. No triangle exists.
2. Exactly one triangle exists.
3. Two triangles exist.

In other words, there may be no solution, one solution, or two solutions. A situation with two solutions is called the **ambiguous case.** Suppose you know the measures of a, b, and A. Consider the following cases.

Case 1: $A < 90°$			
	$a < b \sin A$	$a = b \sin A$	$a > b \sin A$
$a < b$	 no solution	 one solution	 two solutions
$a \geq b$		 one solution	

Case 2: $A \geq 90°$	
$a \leq b$  no solution	$a > b$  one solution

Example 1 **Determine the number of possible solutions for each triangle.**

a. $A = 30°, a = 8, b = 10$

Since $30° < 90°$, consider Case I.

$b \sin A = 10 \sin 30°$
$b \sin A = 10(0.5)$
$b \sin A = 5$

Since $5 < 8 < 10$, there are two solutions for the triangle.

b. $b = 8, c = 10, B = 118°$

Since $118° \geq 90°$, consider Case II.

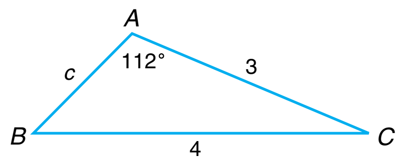
In this triangle, $8 \leq 10$, so there are no solutions.

Once you have determined that there are one or two solutions for a triangle given the measures of two sides and a nonincluded angle, you can use the Law of Sines to solve the triangle.

Example 2 **Find all solutions for each triangle. If no solutions exist, write *none*.**

a. $a = 4, b = 3, A = 112°$

Since $112° \geq 90°$, consider Case II. In this triangle, $4 > 3$, so there is one solution. First, use the Law of Sines to find B.



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{4}{\sin 112°} = \frac{3}{\sin B}$$
$$\sin B = \frac{3 \sin 112°}{4}$$
$$B = \sin^{-1}\left(\frac{3 \sin 112°}{4}\right)$$
$$B \approx 44.05813517$$ *Use a calculator.*

So, $B \approx 44.1°$.

Use the value of B to find C and c.

$C \approx 180° - (112° + 44.1°)$ or about $23.9°$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{4}{\sin 112°} \approx \frac{c}{\sin 23.9°}$$
$$c \approx \frac{4 \sin 23.9°}{\sin 112°}$$
$$c \approx 1.747837108$$ *Use a calculator.*

The solution of this triangle is $B \approx 44.1°$, $C \approx 23.9°$, and $c \approx 1.7$.

b. $A = 51°, a = 40, c = 50$

Since $51° < 90°$, consider Case I.

$c \sin A = 50 \sin 51°$

≈ 38.85729807 *Use a calculator.*

Since $38.9 < 40 < 50$, there are two solutions for the triangle.

Use the Law of Sines to find C.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{40}{\sin 51°} = \frac{50}{\sin C}$$

$$\sin C = \frac{50 \sin 51°}{40}$$

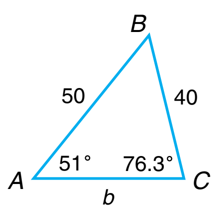
$$C = \sin^{-1}\left(\frac{50 \sin 51°}{40}\right)$$

$C \approx 76.27180414$ *Use a calculator.*

Notice that the sum of the two measures for C is 180°.

So, $C \approx 76.3°$. Since we know there are two solutions, there must be another possible measurement for C. In the second case, C must be less than 180° and have the same sine value. Since we know that if $\alpha < 90$, $\sin \alpha = \sin (180 - \alpha)$, $180° - 76.3°$ or $103.7°$ is another possible measure for C. Now solve the triangle for each possible measure of C.

Solution I



$B \approx 180° - (51° + 76.3°)$
$B \approx 52.7°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

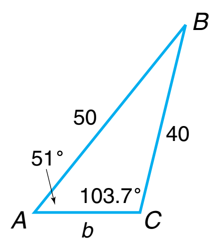
$$\frac{40}{\sin 51°} \approx \frac{b}{\sin 52.7°}$$

$$b \approx \frac{40 \sin 52.7°}{\sin 51°}$$

$b \approx 40.94332444$

One solution is $B \approx 52.7°$, $C \approx 76.3°$, and $b \approx 40.9$.

Solution II



$B \approx 180° - (51° + 103.7°)$
$B \approx 25.3°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{40}{\sin 51°} \approx \frac{b}{\sin 25.3°}$$

$$b \approx \frac{40 \sin 25.3°}{\sin 51°}$$

$b \approx 21.99627275$

Another solution is $B \approx 25.3°$, $C \approx 103.7°$, and $b \approx 22.0$.

GRAPHING CALCULATOR EXPLORATION

You can store values in your calculator and use these values in other computations. In solving triangles, you can store a value for a missing part of the triangle and then use this value when solving for the other missing parts.

TRY THESE

1. Rework Example 2a. Use a calculator to solve for B and store this value. Use the stored value to solve for C and c. Round the answers to the nearest tenth after you have completed all computations.
2. Rework Example 2b. Use a calculator to solve for each possible value for C. Store these values. Use the stored values to solve for B and b in each possible triangle. Round the answers to the nearest tenth after you have completed all computations.

WHAT DO YOU THINK?

3. Compare your answers with those in the examples.
4. Why do you think your answers may vary slightly from those of classmates or the textbook?

Example 3 Real World Application

TOURISM **Refer to the application at the beginning of the lesson. How far along the interstate can the passengers in a car hear the broadcast?**

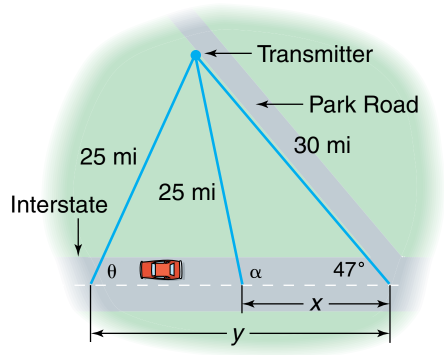
Consider Case 1 because $47° < 90°$. Since $30 \sin 47° \approx 21.9$ and $21.9 < 25 < 30$, there are two triangles with sides 25 miles and 30 miles long and a nonincluded angle of 47°. θ and α represent the two possible angle measures.

$$\frac{25}{\sin 47°} = \frac{30}{\sin \theta}$$

$$\sin \theta = \frac{30 \sin 47°}{25}$$

$$\theta = \sin^{-1}\left(\frac{30 \sin 47°}{25}\right)$$

$$\theta \approx 61.3571157 \quad \textit{Use a calculator.}$$



So, $\theta \approx 61.4°$ and $\alpha \approx 180° - 61.4°$ or 118.6°.

$$\frac{25}{\sin 47°} \approx \frac{y}{\sin(180° - (47° + 61.4°))} \qquad \frac{25}{\sin 47°} \approx \frac{x}{\sin(180° - (47° + 118.6°))}$$

$$\frac{25}{\sin 47°} \approx \frac{y}{\sin 71.6°} \qquad \frac{25}{\sin 47°} \approx \frac{x}{\sin 14.4°}$$

$$y \approx \frac{25 \sin 71.6°}{\sin 47°} \qquad x \approx \frac{25 \sin 14.4°}{\sin 47°}$$

$$y \approx 32.4356057 \qquad x \approx 8.5010128$$

So, $y \approx 32.4$ and $x \approx 8.5$. The passengers can hear the broadcast when the distance to the transmitter is 25 miles or less. So they could hear for about $32.4 - 8.5$ or 23.9 miles along the interstate.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the conditions where the Law of Sines indicates that a triangle cannot exist.
2. **Draw** two triangles where $A = 30°$, $a = 6$, and $b = 10$. Calculate and label the degree measure of each angle rounded to the nearest tenth.
3. **Write** the steps needed to solve a triangle if $A = 120°$, $a = 28$, and $b = 17$.

Guided Practice

Determine the number of possible solutions for each triangle.

4. $A = 113°, a = 15, b = 8$
5. $B = 44°, a = 23, b = 12$

Find all solutions for each triangle. If no solutions exist, write *none.* Round to the nearest tenth.

6. $C = 17°, a = 10, c = 11$
7. $A = 140°, b = 10, a = 3$
8. $A = 38°, b = 10, a = 8$
9. $C = 130°, c = 17, b = 5$

10. **Communications** A vertical radio tower is located on the top of a hill that has an angle of elevation of 10°. A 70-foot guy wire is attached to the tower 45 feet above the hill.
    a. Make a drawing to illustrate the situation.
    b. What angle does the guy wire make with the side of the hill?
    c. How far from the base of the tower is the guy wire anchored to the hill?

EXERCISES

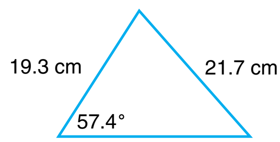
Practice

Determine the number of possible solutions for each triangle.

11. $A = 57°, a = 11, b = 19$
12. $A = 30°, a = 13, c = 26$
13. $B = 61°, a = 12, b = 8$
14. $A = 58°, C = 94°, b = 17$
15. $C = 100°, a = 18, c = 15$
16. $B = 37°, a = 32, b = 27$
17. If $A = 65°$, $a = 55$, and $b = 57$, how many possible values are there for B?

Find all solutions for each triangle. If no solutions exist, write *none.* Round to the nearest tenth.

18. $a = 6, b = 8, A = 150°$
19. $a = 26, b = 29, A = 58°$
20. $A = 30°, a = 4, b = 8$
21. $C = 70°, c = 24, a = 25$
22. $A = 40°, B = 60°, c = 20$
23. $a = 14, b = 12, B = 90°$
24. $B = 36°, b = 19, c = 30$
25. $A = 107.2°, a = 17.2, c = 12.2$
26. $A = 76°, a = 5, b = 20$
27. $C = 47°, a = 10, c = 16$
28. $B = 40°, b = 42, c = 60$
29. $b = 40, a = 32, A = 125.3°$
30. Copy the triangle at the right and label all measurements of the triangle.



www.amc.glencoe.com/self_check_quiz

CONTENTS

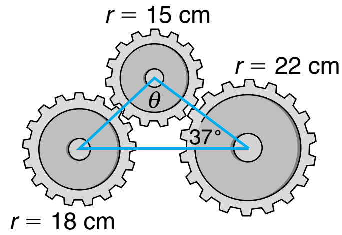
31. Find the perimeter of each of the two noncongruent triangles where $a = 15$, $b = 20$ and $A = 29°$.

32. There are two noncongruent triangles where $B = 55°$, $a = 15$, and $b = 13$. Find the measures of the angles of the triangle with the greater perimeter.

Applications and Problem Solving

Real World Application

33. **Gears** An engineer designed three gears as shown at the right. What is the measure of θ?



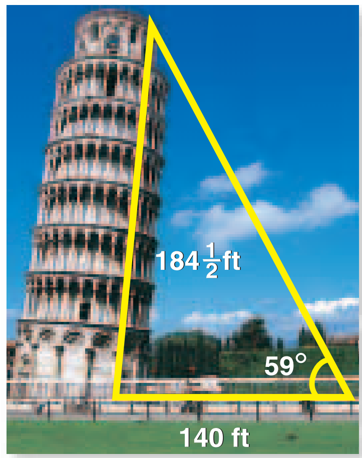
34. **Critical Thinking** If $b = 14$ and $A = 30°$, determine the possible values of a for each situation.

a. The triangle has no solutions.

b. The triangle has one solution.

c. The triangle has two solutions.

35. **Architecture** The original height of the Leaning Tower of Pisa was $184\frac{1}{2}$ feet. At a distance of 140 feet from the base of the tower, the angle of elevation from the ground to the top of the tower is 59°. How far is the tower leaning from the original vertical position?



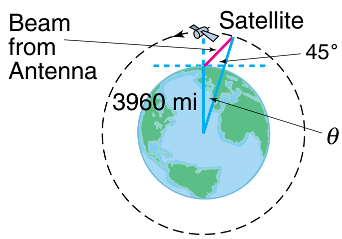
36. **Navigation** The captain of the Coast Guard Cutter Pendant plans to sail to a port that is 450 miles away and 12° east of north. The captain first sails the ship due north to check a buoy. He then turns the ship and sails 316 miles to the port.

a. In what direction should the captain turn the ship to arrive at the port?

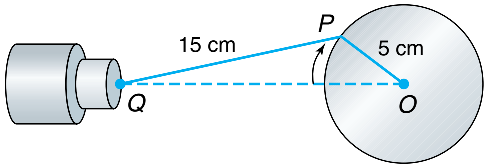
b. How many hours will it take to arrive at the turning point if the captain chooses a speed of 23 miles per hour?

c. Instead of the plan above, the captain decides to sail 200 miles north, turn through an angle of 20° east of north, and then sail along a straight course. Will the ship reach the port by following this plan?

37. **Communications** A satellite is orbiting Earth every 2 hours. The satellite is directly over a tracking station which has its antenna aimed 45° above the horizon. The satellite is orbiting 1240 miles above Earth, and the radius of Earth is about 3960 miles. How long ago did the satellite pass through the beam of the antenna? (*Hint*: First calculate θ.)



38. **Mechanics** The blades of a power lawn mower are rotated by a two-stroke engine with a piston sliding back and forth in the engine cylinder. As the piston moves back and forth, the connecting rod rotates the circular crankshaft. Suppose the crankshaft is 5 centimeters long and the connecting rod is 15 centimeters. If the crankshaft rotates 20 revolutions per second and P is at the horizontal position when it begins to rotate, how far is the piston from the rim of the crankshaft after 0.01 second?



39. **Critical Thinking** If $b = 12$ and $c = 17$, find the values of B for each situation.

   a. The triangle has no solutions.

   b. The triangle has one solution.

   c. The triangle has two solutions.

Mixed Review

40. **Geometry** Determine the area of a rhombus if the length of a side is 24 inches and one of its angles is 32°. *(Lesson 5-6)*

41. **Fire Fighting** A fire is sighted from a fire tower in Wayne National Forest in Ohio. The ranger found that the angle of depression to the fire is 22°. If the tower is 75 meters tall, how far is the fire from the base of the tower? *(Lesson 5-4)*

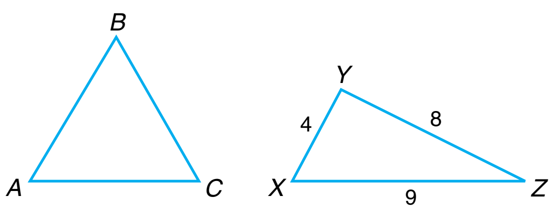
42. State the number of roots of the equation $4x^3 - 4x^2 + 13x - 6 = 0$. Then solve the equation. *(Lesson 4-2)*

43. Determine whether the functions $y = \frac{3x}{x - 1}$ and $y = \frac{x + 1}{3x}$ are inverses of one another. Explain. *(Lesson 3-4)*

44. Solve the system of equations algebraically. *(Lesson 2-1)*
$5x - 2y = 9$
$y = 3x - 1$

45. Write the standard form of the equation whose graph is perpendicular to the graph of $-2x + 5y = 7$ and passes through the point at $(-6, 4)$. *(Lesson 1-5)*

46. **SAT Practice** **Grid-In** For the triangles shown below, the perimeter of $\triangle ABC$ equals the perimeter of $\triangle XYZ$. If $\triangle ABC$ is equilateral, what is the length of $\overline{AB}$?



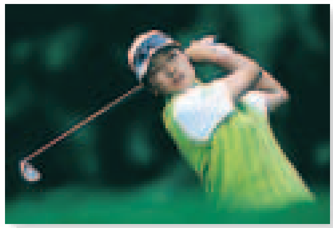
Extra Practice See p. A35.

5-8 The Law of Cosines

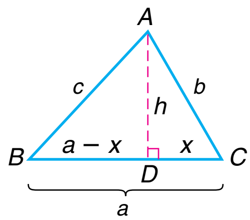
OBJECTIVES
- Solve triangles by using the Law of Cosines.
- Find the area of triangles if the measures of the three sides are given.

Real World Application

GOLF For a right-handed golfer, a *slice* is a shot that curves to the right of its intended path, and a *hook* curves off to the left. Suppose Se Ri Pak hits the ball from the seventh tee at the U.S. Women's Open and the shot is a 160-yard slice 4° from the path straight to the cup. If the tee is 177 yards from the cup, how far does the ball lie from the cup? *This problem will be solved in Example 1.*



From geometry, you know that a unique triangle can be formed if the measures of three sides of a triangle are known and the sum of any two measures is greater than the remaining measure. You also know that a unique triangle can be formed if the measures of two sides and an included angle are known. However, the Law of Sines cannot be used to solve these triangles. Another formula is needed. Consider $\triangle ABC$ with a height of h units and sides measuring a units, b units, and c units. Suppose $\overline{DC}$ is x units long. Then $\overline{BD}$ is $(a - x)$ units long.



The Pythagorean Theorem and the definition of the cosine ratio can be used to show how $\angle C$, a, b, and c are related.

$c^2 = (a - x)^2 + h^2$	*Apply the Pythagorean Theorem for $\triangle ADB$.*
$c^2 = a^2 - 2ax + x^2 + h^2$	*Expand $(a - x)^2$.*
$c^2 = a^2 - 2ax + b^2$	*$b^2 = x^2 + h^2$ in $\triangle ADC$.*
$c^2 = a^2 - 2a(b \cos C) + b^2$	*$\cos C = \frac{x}{b}$, so $b \cos C = x$.*
$c^2 = a^2 + b^2 - 2ab \cos C$	*Simplify.*

By drawing altitudes from B and C, you can derive similar formulas for a^2 and b^2. All three formulas, which make up the **Law of Cosines,** can be summarized as follows.

Law of Cosines

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C, respectively. Then, the following are true.

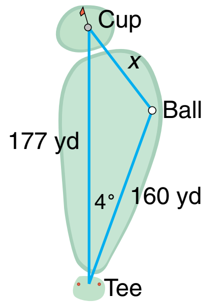
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

You can use the Law of Cosines to solve the application at the beginning of the lesson.

Example 1 Real World Application

GOLF **Refer to the application at the beginning of the lesson. How far does the ball lie from the cup?**

In this problem, you know the measurements of two sides of a triangle and the included angle. Use the Law of Cosines to find the measure of the third side of the triangle.



$x^2 = 177^2 + 160^2 - 2(177)(160) \cos 4°$

$x^2 \approx 426.9721933$ *Use a calculator.*

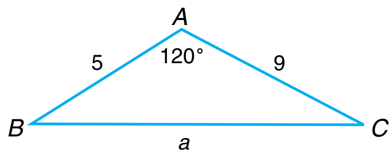
$x \approx 20.66330548$

The ball is about 20.7 yards from the cup.

Many times, you will have to use both the Law of Cosines and the Law of Sines to solve triangles.

Example 2 **Solve each triangle.**

a. $A = 120°, b = 9, c = 5$



Graphing Calculator Tip

If you store the calculated value of *a* in your calculator, your solution will differ slightly from the one using the rounded value of *a*.

$a^2 = b^2 + c^2 - 2bc \cos A$ *Law of Cosines*

$a^2 = 9^2 + 5^2 - 2(9)(5) \cos 120°$ *Substitute 9 for b, 5 for c, and 120° for A.*

$a^2 = 151$ *Use a calculator.*

$a \approx 12.28820573$

So, $a \approx 12.3$.

$\frac{a}{\sin A} = \frac{b}{\sin B}$ *Law of Sines*

$\frac{12.3}{\sin 120°} \approx \frac{9}{\sin B}$ *Substitute 12.3 for a, 9 for b, and 120° for A.*

$\sin B \approx \frac{9 \sin 120°}{12.3}$

$B \approx \sin^{-1}\left(\frac{9 \sin 120°}{12.3}\right)$

$B \approx 39.32193819$ *Use a calculator.*

So, $B \approx 39.3°$.

$C \approx 180° - (120° + 39.3°)$

$C \approx 20.7°$

The solution of this triangle is $a \approx 12.3$, $B \approx 39.3°$, and $C \approx 20.7°$.

b. $a = 24, b = 40, c = 18$

Recall that θ and $180° - \theta$ have the same sine function value, but different cosine function values. Therefore, a good strategy to use when all three sides are given is to use the Law of Cosines to determine the measure of the possible obtuse angle first. Since b is the longest side, B is the angle with the greatest measure, and therefore a possible obtuse angle.

$$b^2 = a^2 + c^2 - 2ac \cos B \quad \textit{Law of Cosines}$$
$$40^2 = 24^2 + 18^2 - 2(24)(18) \cos B$$
$$\frac{40^2 - 24^2 - 18^2}{-2(24)(18)} = \cos B$$
$$\cos^{-1}\left(\frac{40^2 - 24^2 - 18^2}{-2(24)(18)}\right) = B$$
$$144.1140285 \approx B \quad \textit{Use a calculator.}$$

So, $B \approx 144.1°$.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \textit{Law of Sines}$$
$$\frac{24}{\sin A} \approx \frac{40}{\sin 144.1°}$$
$$\sin A \approx \frac{24 \sin 144.1°}{40}$$
$$A \approx \sin^{-1}\left(\frac{24 \sin 144.1°}{40}\right)$$
$$A \approx 20.59888389 \quad \textit{Use a calculator.}$$

So, $A \approx 20.6°$.

$$C \approx 180 - (20.6 + 144.1)$$
$$C \approx 15.3$$

The solution of this triangle is $A \approx 20.6°$, $B \approx 144.1°$, and $C \approx 15.3°$.

If you know the measures of three sides of a triangle, you can find the area of the triangle by using the Law of Cosines and the formula $K = \frac{1}{2}bc \sin A$.

Example 3 **Find the area of $\triangle ABC$ if $a = 4$, $b = 7$, and $c = 9$.**

First, solve for A by using the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$4^2 = 7^2 + 9^2 - 2(7)(9) \cos A$$
$$\frac{4^2 - 7^2 - 9^2}{-2(7)(9)} = \cos A$$
$$\cos^{-1}\left(\frac{4^2 - 7^2 - 9^2}{-2(7)(9)}\right) = A \quad \textit{Definition of } \cos^{-1}$$
$$25.2087653 \approx A \quad \textit{Use a calculator.}$$

So, $A \approx 25.2°$.

Then, find the area.

$$K = \frac{1}{2}bc \sin A$$
$$K \approx \frac{1}{2}(7)(9) \sin 25.2°$$
$$K \approx 13.41204768$$

The area of the triangle is about 13.4 square units.

If you know the measures of three sides of any triangle, you can also use **Hero's Formula** to find the area of the triangle.

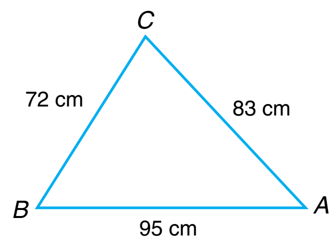
Hero's Formula

If the measures of the sides of a triangle are a, b, and c, then the area, K, of the triangle is found as follows.

$$K = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{1}{2}(a+b+c)$$

s is called the semiperimeter.

Example 4 **Find the area of $\triangle ABC$. Round to the nearest tenth.**



First, find the semiperimeter of $\triangle ABC$.

$s = \frac{1}{2}(a + b + c)$

$s = \frac{1}{2}(72 + 83 + 95)$

$s = 125$

Now, apply Hero's Formula.

$K = \sqrt{s(s-a)(s-b)(s-c)}$

$K = \sqrt{125(125-72)(125-83)(125-95)}$

$K = \sqrt{8{,}347{,}500}$

$K \approx 2889.204043$ *Use a calculator.*

The area of the triangle is about 2889.2 square centimeters.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** two situations where the Law of Cosines is needed to solve a triangle.
2. **Give an example** of three measurements of sides that do not form a triangle.
3. **Explain** how the Pythagorean Theorem is a special case of the Law of Cosines.
4. *Math Journal* **Draw and label** a right triangle that can be solved using the trigonometric functions. Draw and label a triangle that can be solved using the Law of Sines, but not the Law of Cosines. Draw and label a triangle that can be solved using the Law of Cosines. Solve each triangle.

Guided Practice

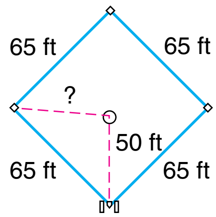
Solve each triangle. Round to the nearest tenth.

5. $a = 32$, $b = 38$, $c = 46$
6. $a = 25$, $b = 30$, $C = 160°$
7. The sides of a triangle are 18 inches, 21 inches, and 14 inches. Find the measure of the angle with the greatest measure.

Find the area of each triangle. Round to the nearest tenth.

8. $a = 2$, $b = 7$, $c = 8$
9. $a = 25$, $b = 13$, $c = 17$

10. **Softball** In slow-pitch softball, the diamond is a square that is 65 feet on each side. The distance between the pitcher's mound and home plate is 50 feet. How far does the pitcher have to throw the softball from the pitcher's mound to third base to stop a player who is trying to steal third base?



EXERCISES

Practice

Solve each triangle. Round to the nearest tenth.

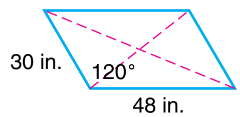
11. $b = 7, c = 10, A = 51°$
12. $a = 5, b = 6, c = 7$
13. $a = 4, b = 5, c = 7$
14. $a = 16, c = 12, B = 63°$
15. $a = 11.4, b = 13.7, c = 12.2$
16. $C = 79.3°, a = 21.5, b = 13$
17. The sides of a triangle measure 14.9 centimeters, 23.8 centimeters, and 36.9 centimeters. Find the measure of the angle with the least measure.
18. **Geometry** Two sides of a parallelogram measure 60 centimeters and 40 centimeters. If one angle of the parallelogram measures 132°, find the length of each diagonal.

interNET CONNECTION

Graphing Calculator Programs
For a graphing calculator program that determines the area of a triangle, given the lengths of all sides of the triangle, visit **www.amc.glencoe.com**

Find the area of each triangle. Round to the nearest tenth.

19. $a = 4, b = 6, c = 8$
20. $a = 17, b = 13, c = 19$
21. $a = 20, b = 30, c = 40$
22. $a = 33, b = 51, c = 42$
23. $a = 174, b = 138, c = 188$
24. $a = 11.5, b = 13.7, c = 12.2$
25. **Geometry** The lengths of two sides of a parallelogram are 48 inches and 30 inches. One angle measures 120°.
    a. Find the length of the longer diagonal.
    b. Find the area of the parallelogram.

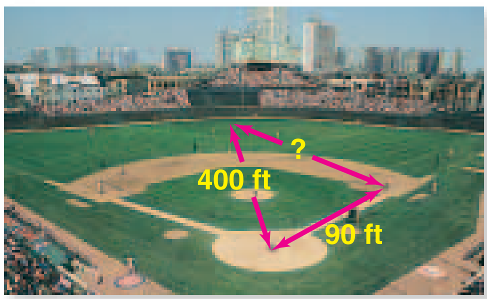


26. **Geometry** The side of a rhombus is 15 centimeters long, and the length of its longer diagonal is 24.6 centimeters.
    a. Find the area of the rhombus.
    b. Find the measures of the angles of the rhombus.

Applications and Problem Solving

Real World Application

27. **Baseball** In baseball, dead center field is the farthest point in the outfield on the straight line through home plate and second base. The distance between consecutive bases is 90 feet. In Wrigley Field in Chicago, dead center field is 400 feet from home plate. How far is dead center field from first base?



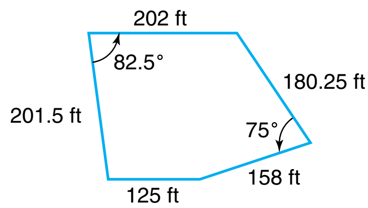
28. **Critical Thinking** The lengths of the sides of a triangle are 74 feet, 38 feet, and 88 feet. What is the length of the altitude drawn to the longest side?

29. Air Travel The distance between Miami and Orlando is about 220 miles. A pilot flying from Miami to Orlando starts the flight 10° off course to avoid a storm.

a. After flying in this direction for 100 miles, how far is the plane from Orlando?

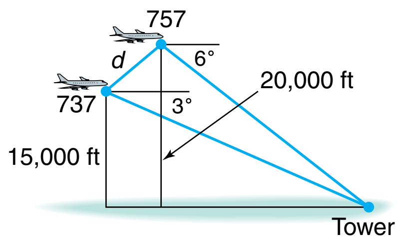
b. If the pilot adjusts his course after 100 miles, how much farther is the flight than a direct route?

30. Critical Thinking Find the area of the pentagon at the right.



31. Soccer A soccer player is standing 35 feet from one post of the goal and 40 feet from the other post. Another soccer player is standing 30 feet from one post of the same goal and 20 feet from the other post. If the goal is 24 feet wide, which player has a greater angle to make a shot on goal?

32. Air Traffic Control A 757 passenger jet and a 737 passenger jet are on their final approaches to San Diego International Airport.



a. The 757 is 20,000 feet from the ground, and the angle of depression to the tower is 6°. Find the distance between the 757 and the tower.

b. The 737 is 15,000 feet from the ground, and the angle of depression to the tower is 3°. What is the distance between the 737 and the tower?

c. How far apart are the jets?

Mixed Review

33. Determine the number of possible solutions for $\triangle ABC$ if $A = 63.2^\circ$, $b = 18$ and $a = 17$. *(Lesson 5-7)*

34. Landmarks The San Jacinto Column in Texas is 570 feet tall and, at a particular time, casts a shadow 700 feet long. Find the angle of elevation to the sun at that time. *(Lesson 5-5)*

35. Find the reference angle for -775°. *(Lesson 5-1)*

36. Find the value of k so that the remainder of $(x^3 - 7x^2 - kx + 6)$ divided by $(x - 3)$ is 0. *(Lesson 4-3)*

37. Find the slope of the line through points at $(2t, t)$ and $(5t, 5t)$. *(Lesson 1-3)*

38. SAT/ACT Practice Find an expression equivalent to $\left(\frac{2x^2}{y}\right)^3$.

A $\frac{8x^6}{y^3}$ **B** $\frac{64x^6}{y^3}$ **C** $\frac{6x^5}{y^3}$ **D** $\frac{8x^5}{y^3}$ **E** $\frac{2x^5}{y^4}$

Extra Practice See p. A35.

GRAPHING CALCULATOR EXPLORATION

5-8B Solving Triangles

An Extension of Lesson 5-8

OBJECTIVE

- Use a program to solve triangles.

The following program allows you to enter the coordinates of the vertices of a triangle in the coordinate plane and display as output the side lengths and the angle measures in degrees.

```
PROGRAM: SOLVTRI
: Disp "INPUT VERTEX A"
: Input A: Input P
: Disp "INPUT VERTEX B"
: Input B: Input Q
: Disp "INPUT VERTEX C"
: Input C: Input S
: √((A − B)² + (P − Q)²) → E
: √((B − C)² + (Q − S)²) → F
: √((A − C)² + (P − S)²) → G
: cos⁻¹ ((F² − E² − G²)/(−2EG)) → M
: cos⁻¹ ((G² − E² − F²)/(−2EF)) → N
: cos⁻¹ ((E² − F² − G²)/(−2FG)) → O
: Disp "SIDE AB" : Disp E
: Disp "SIDE BC" : Disp F
: Disp "SIDE AC" : Disp G
: Pause
: Disp "ANGLE A" : Disp M
: Disp "ANGLE B" : Disp N
: Disp "ANGLE C" : Disp O
: Stop
```

- To solve a triangle using the program, you first need to place the triangle in a coordinate plane, determine the coordinates of the vertices, and then input the coordinates when prompted by the program.

- When you place the triangle in the coordinate plane, it is a good idea to choose a side whose length is given and place that side on the positive x-axis so that its left endpoint is at (0, 0). You can use the length of the side to determine the coordinates of the other endpoint.

- To locate the third vertex, you can use the given information about the triangle to write equations whose graphs intersect at the third vertex. Graph the equations and use intersect on the **CALC** menu to find the coordinates of the third vertex.

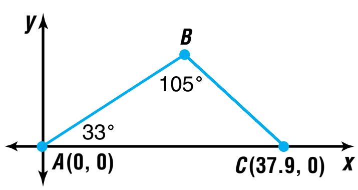
- You are now ready to run the program.

interNET CONNECTION

You can download this program by visiting our Web site at **www.amc.glencoe.com**

Example 1 **Use the program to solve $\triangle ABC$ if $A = 33°$, $B = 105°$, and $b = 37.9$.**

Before you use the calculator, do some advance planning. Make a sketch of how you can place the triangle in the coordinate plane to have the third vertex above the x-axis. One possibility is shown at the right.



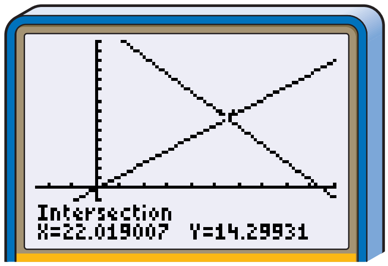
(continued on the next page)

The slope of $\overline{AB}$ is $\frac{y-0}{x-0}$ or $\frac{y}{x}$. This is tan 33°. So, $\frac{y}{x} = \tan 33°$ or $y = (\tan 33°)x$.

The slope of $\overline{BC}$ is $\frac{y-0}{x-37.9}$. This is tangent of $(180° - 42°)$.

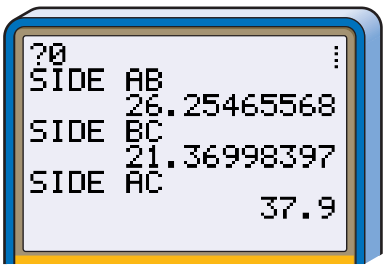
So, $\tan 138° = \frac{y}{x-37.9}$ or $y = (\tan 138°)(x - 37.9)$.

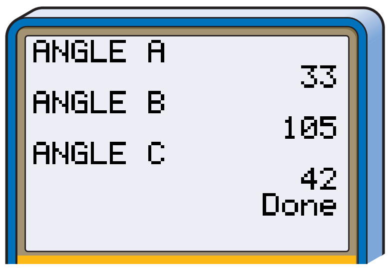
Enter the equations (without the degree signs) on the **Y=** list, and select appropriate window settings. Graph the equations and use intersect in the **CALC** menu to find the coordinates of vertex *B*. When the calculator displays the coordinates at the bottom of the screen, go immediately to the program. Be sure you do nothing to alter the values of *x* and *y* that were displayed.



[-10, 40] scl: 5 by [-10, 40] scl: 5

When the program prompts you to input vertex *A*, enter 0 and 0. For vertex *B*, press [X,T,θ,*n*] [ENTER] [ALPHA] **[Y]** [ENTER]. For vertex *C*, enter the numbers 37.9 and 0.The calculator will display the side lengths of the triangle and pause. To display the angle measures, press [ENTER].





The side lengths and angle measures agree with those in the text. Therefore, $a \approx 21.4$, $c \approx 26.3$, and $C = 42°$. *Compare the results with those in Example 1 of Lesson 5-6, on page 314.*

TRY THESE

1. Solve $\triangle ABC$ given that $AC = 6$, $BC = 8$, and $A = 35°$. (*Hint*: Place $\overline{AC}$ so that the endpoints have coordinates (0, 0) and (6, 0). Use the graphs of $y = (\tan 35°)x$ and $y = \sqrt{5^2 - (x-6)^2}$ to determine the coordinates of vertex *B*.)
2. Use **SOLVTRI** to solve the triangles in Example 2b of Lesson 5-7.

WHAT DO YOU THINK?

3. What law about triangles is the basis of the program **SOLVTRI**?
4. Suppose you want to use **SOLVTRI** to solve the triangle in Example 2a of Lesson 5-7. How would you place the triangle in the coordinate plane?

VOCABULARY

ambiguous case (p. 320)
angle of depression (p. 300)
angle of elevation (p. 300)
apothem (p. 300)
arccosine relation (p. 305)
arcsine relation (p. 305)
arctangent relation (p. 305)
circular function (p. 292)
cofunctions (p. 287)
cosecant (pp. 286, 292)
cosine (pp. 285, 291)
cotangent (pp. 286, 292)
coterminal angles (p. 279)
degree (p. 277)
Hero's Formula (p. 330)
hypotenuse (p. 284)
initial side (p. 277)
inverse (p. 305)
Law of Cosines (p. 327)
Law of Sines (p. 313)
leg (p. 284)
minute (p. 277)
quadrantal angle (p. 278)
reference angle (p. 280)
secant (pp. 286, 292)
second (p. 277)
side adjacent (p. 284)
side opposite (p. 284)
sine (pp. 285, 291)
solve a triangle (p. 307)
standard position (p. 277)
tangent (pp. 285, 292)
terminal side (p. 277)
trigonometric function (p. 292)
trigonometric ratio (p. 285)
unit circle (p. 291)
vertex (p. 277)

UNDERSTANDING AND USING THE VOCABULARY

State whether each sentence is true or false. If false, replace the underlined word(s) to make a true statement.

1. An angle of elevation is the angle between a horizontal line and the line of sight from the observer to an object at a lower level.
2. The inverse of the cosine function is the arcsine relation.
3. A degree is subdivided into 60 equivalent parts known as minutes.
4. The leg that is a side of an acute angle of a right triangle is called the side opposite the angle.
5. If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, the relations $\cos\theta = x$ and $\sin\theta = y$ are called circular functions.
6. Two angles in standard position are called reference angles if they have the same terminal side.
7. Trigonometric ratios are defined by the ratios of right triangles.
8. The Law of Sines is derived from the Pythagorean Theorem.
9. The ray that rotates to form an angle is called the initial side.
10. A circle of radius 1 is called a unit circle.

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 5-1 Identify angles that are coterminal with a given angle.

- If a 585° angle is in standard position, determine a coterminal angle that is between 0° and 360°. State the quadrant in which the terminal side lies.

  First, determine the number of complete rotations (k) by dividing 585 by 360.

  $\frac{585}{360} = 1.625$

  Use $\alpha + 360k°$ to find the value of α.

  $\alpha + 360(1)° = 585°$

  $\alpha = 225°$

  The coterminal angle (α) is 225°. Its terminal side lies in the third quadrant.

REVIEW EXERCISES

Change each measure to degrees, minutes, and seconds.

11. 57.15° **12.** −17.125°

If each angle is in standard position, determine a coterminal angle that is between 0° and 360°. State the quadrant in which the terminal side lies.

13. 860° **14.** 1146°

15. −156° **16.** 998°

17. −300° **18.** 1072°

19. 654° **20.** −832°

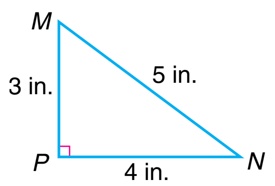
Find the measure of the reference angle for each angle.

21. −284° **22.** 592°

OBJECTIVES AND EXAMPLES

Lesson 5-2 Find the values of trigonometric ratios for acute angles of right triangles.

- Find the values of the six trigonometric ratios for $\angle M$.



$\sin M = \frac{4}{5}$ $\quad$ *sin M = side opposite / hypotenuse*

$\cos M = \frac{3}{5}$ $\quad$ *cos M = side adjacent / hypotenuse*

$\tan M = \frac{4}{3}$ $\quad$ *tan M = side opposite / side adjacent*

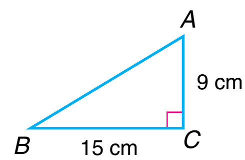
$\csc M = \frac{5}{4}$ $\quad$ *csc M = hypotenuse / side opposite*

$\sec M = \frac{5}{3}$ $\quad$ *sec M = hypotenuse / side adjacent*

$\cot M = \frac{3}{4}$ $\quad$ *cot M = side adjacent / side opposite*

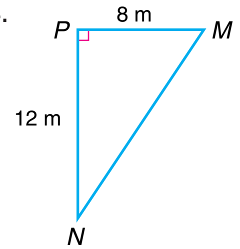
REVIEW EXERCISES

23. Find the values of the sine, cosine, and tangent for $\angle A$.

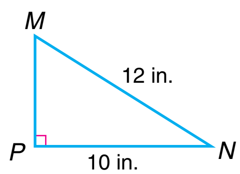


Find the values of the six trigonometric functions for each $\angle M$.

24.



25.



26. If $\sec \theta = \frac{7}{5}$, find $\cos \theta$.

OBJECTIVES AND EXAMPLES

Lesson 5-3 Find the values of the six trigonometric functions of an angle in standard position given a point on its terminal side.

Find the values of the six trigonometric functions for angle θ in standard position if a point with coordinates (3, 4) lies on its terminal side.

$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{25}$ or 5

$\sin\theta = \frac{y}{r} = \frac{4}{5}$ $\quad\cos\theta = \frac{x}{r} = \frac{3}{5}$

$\tan\theta = \frac{y}{x} = \frac{4}{3}$ $\quad\csc\theta = \frac{r}{y} = \frac{5}{4}$

$\sec\theta = \frac{r}{x} = \frac{5}{3}$ $\quad\sin\theta = \frac{x}{y} = \frac{3}{4}$

REVIEW EXERCISES

Find the values of the six trigonometric functions for each angle θ in standard position if a point with the given coordinates lies on its terminal side.

27. (3, 3) **28.** (−5, 12)
29. (8, −2) **30.** (−2, 0)
31. (4, 5) **32.** (−5, −9)
33. (−4, 4) **34.** (5, 0)

Suppose θ is an angle in standard position whose terminal side lies in the given quadrant. For each function, find the values of the remaining five trigonometric functions for θ.

35. $\cos\theta = -\frac{3}{8}$; Quadrant II

36. $\tan\theta = 3$; Quadrant III

Lesson 5-4 Use trigonometry to find the measures of the sides of right triangles.

Refer to $\triangle ABC$ at the right. If $A = 25°$ and $b = 12$, find c.

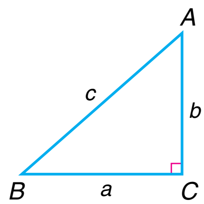
$\cos A = \frac{b}{c}$

$\cos 25° = \frac{12}{c}$

$c = \frac{12}{\cos 25°}$

$c \approx 13.2$

Solve each problem. Round to the nearest tenth.



37. If $B = 42°$ and $c = 15$, find b.
38. If $A = 38°$ and $a = 24$, find c.
39. If $B = 67°$ and $b = 24$, find a.

Lesson 5-5 Solve right triangles.

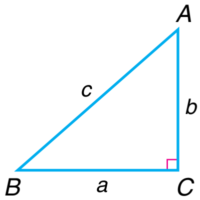
If $c = 10$ and $a = 9$, find A.

$\sin A = \frac{a}{c}$

$\sin A = \frac{9}{10}$

$A = \sin^{-1}\frac{9}{10}$

$A \approx 64.2°$



Solve each equation if $0° \le x \le 360°$.

40. $\tan\theta = \frac{\sqrt{3}}{3}$ **41.** $\cos\theta = -1$

Refer to $\triangle ABC$ at the left. Solve each triangle described. Round to the nearest tenth if necessary.

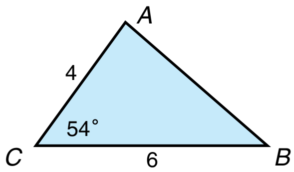
42. $B = 49°$, $a = 16$
43. $b = 15$, $c = 20$
44. $A = 64°$, $c = 28$

OBJECTIVES AND EXAMPLES

Lesson 5-6 Find the area of a triangle if the measures of two sides and the included angle or the measures of two angles and a side are given.

- Find the area of $\triangle ABC$ if $a = 6$, $b = 4$, and $C = 54°$.

  Draw a diagram.

  $K = \frac{1}{2}ab \sin C$

  $K = \frac{1}{2}(6)(4) \sin 54°$

  $K \approx 9.708203932$

  The area of $\triangle ABC$ is about 9.7 square units.

REVIEW EXERCISES

Solve each triangle. Round to the nearest tenth.

45. $B = 70°$, $C = 58°$, $a = 84$

46. $c = 8$, $C = 49°$, $B = 57°$

Find the area of each triangle. Round to the nearest tenth.

47. $A = 20°$, $a = 19$, $C = 64°$

48. $b = 24$, $A = 56°$, $B = 78°$

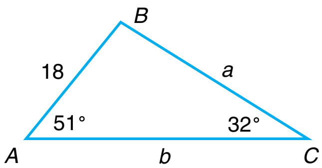
49. $b = 65.5$, $c = 89.4$, $A = 58.2°$

50. $B = 22.6°$, $a = 18.4$, $c = 6.7$

Lesson 5-7 Solve triangles by using the Law of Sines.

- In $\triangle ABC$, if $A = 51°$, $C = 32°$, and $c = 18$, find a.

  Draw a diagram.

  $\frac{a}{\sin A} = \frac{c}{\sin C}$

  $\frac{a}{\sin 51°} = \frac{18}{\sin 32°}$

  $a = \frac{(\sin 51°)18}{\sin 32°}$

  $a \approx 26.4$

Find all solutions for each triangle. If no solutions exist, write *none*. Round to the nearest tenth.

51. $A = 38.7°$, $a = 172$, $c = 203$

52. $a = 12$, $b = 19$, $A = 57°$

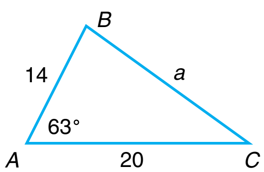
53. $A = 29°$, $a = 12$, $c = 15$

54. $A = 45°$, $a = 83$, $b = 79$

Lesson 5-8 Solve triangles by using the Law of Cosines.

- In $\triangle ABC$, if $A = 63°$, $b = 20$, and $c = 14$, find a.

  Draw a diagram.

  $a^2 = b^2 + c^2 - 2bc \cos A$

  $a^2 = 20^2 + 14^2 - 2(20)(14) \cos 63°$

  $a^2 \approx 341.77$

  $a \approx 18.5$

Solve each triangle. Round to the nearest tenth.

55. $A = 51°$, $b = 40$, $c = 45$

56. $B = 19°$, $a = 51$, $c = 61$

57. $a = 11$, $b = 13$, $c = 20$

58. $B = 24°$, $a = 42$, $c = 6.5$

CONTENTS

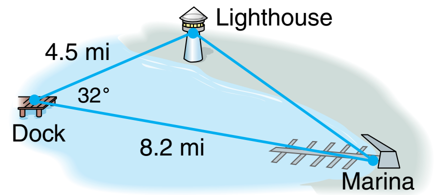
APPLICATIONS AND PROBLEM SOLVING

59. Camping Haloke and his friends are camping in a tent. Each side of the tent forms a right angle with the ground. The tops of two ropes are attached to each side of the tent 8 feet above the ground. The other ends of the two ropes are attached to stakes on the ground. *(Lesson 5-4)*

a. If the rope is 12 feet long, what angle does it make with the level ground?

b. What is the distance between the bottom of the tent and each stake?

60. Navigation Hugo is taking a boat tour of a lake. The route he takes is shown on the map below. *(Lesson 5-8)*



a. How far is it from the lighthouse to the marina?

b. What is the angle between the route from the dock to the lighthouse and the route from the lighthouse to the marina?

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. A triangle has an area of 125 square centimeters and an angle that measures 35°. What are possible lengths of two sides of the triangle?
2. a. Give the lengths of two sides and a nonincluded angle so that no triangle exists. Explain why no triangle exists for the measures you give.

   b. Can you change the length of one of the sides you gave in part a so that two triangles exist? Explain.

PORTFOLIO

Explain how you can find the area of a triangle when you know the length of all three sides of the triangle.

Additional Assessment See p. A60 for Chapter 5 practice test.

Unit 2 interNET Project

THE CYBERCLASSROOM

Does anybody out there know anything about trigonometry?

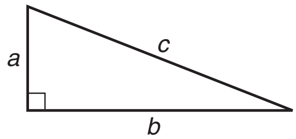
- Search the Internet to find at least three web sites that offer lessons on trigonometry. Some possible sites are actual mathematics courses offered on the Internet or webpages designed by teachers.
- Compare the Internet lessons with the lessons from this chapter. Note any similarities or differences.
- Select one topic from Chapter 5. Combine the information from your textbook and the lessons you found on the Internet. Write a summary of this topic using all the information you have gathered.

CHAPTER 5 SAT & ACT Preparation

THE PRINCETON REVIEW

Pythagorean Theorem

All SAT and ACT tests contain several problems that you can solve using the Pythagorean Theorem. The **Pythagorean Theorem** states that in a right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.



$$a^2 + b^2 = c^2$$

TEST-TAKING TIP

The 3-4-5 right triangle and its multiples like 6-8-10 and 9-12-15 occur frequently on the SAT and ACT. Other commonly used Pythagorean triples include 5-12-13 and 7-24-25. Memorize them.

SAT EXAMPLE

1. A 25-foot ladder is placed against a vertical wall of a building with the bottom of the ladder standing on concrete 7 feet from the base of the building. If the top of the ladder slips down 4 feet, then the bottom of the ladder will slide out how many feet?

A 4 ft

B 5 ft

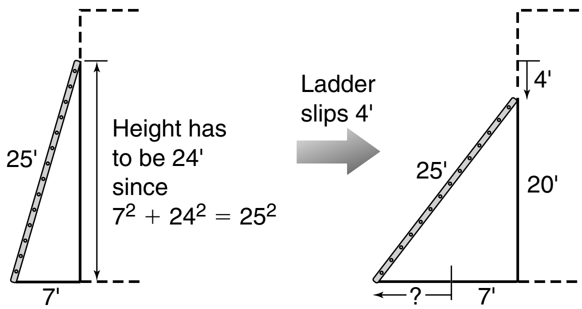
C 6 ft

D 7 ft

E 8 ft

HINT This problem does not have a diagram. So, start by drawing diagrams.

Solution The ladder placed against the wall forms a 7-24-25 right triangle. After the ladder slips down 4 feet, the new right triangle has sides that are multiples of a 3-4-5 right triangle, 15-20-25.



The ladder is now 15 feet from the wall. This means the ladder slipped 15 − 7 or 8 feet. The correct answer is choice **E.**

ACT EXAMPLE

2. In the figure below, right triangles *ABC* and *ACD* are drawn as shown. If $AB = 20$, $BC = 15$, and $AD = 7$, then $CD = ?$

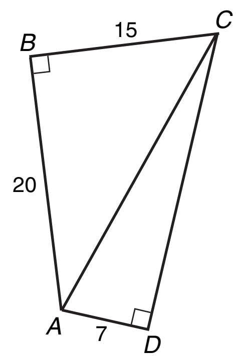
A 21

B 22

C 23

D 24

E 25



HINT Be on the lookout for problems like this one in which the application of the Pythagorean Theorem is not obvious.

Solution Notice that quadrilateral *ABCD* is separated into two right triangles, $\triangle ABC$ and $\triangle ADC$.

$\triangle ABC$ is a 15-20-25 right triangle (a multiple of the 3-4-5 right triangle). So, side $\overline{AC}$ (the hypotenuse) is 25 units long.

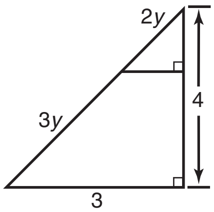
$\overline{AC}$ is also the hypotenuse of $\triangle ADC$. So, $\triangle ADC$ is a 7-24-25 right triangle.

Therefore, $\overline{CD}$ is 24 units long. The correct answer is choice **D.**

After working each problem, record the correct answer on the answer sheet provided or use your own paper.

Multiple Choice

1. In the figure below, $y =$



A 1 **B** 2 **C** 3 **D** 4 **E** 5

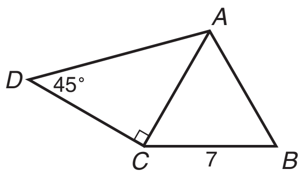
2. What graph would be created if the equation $x^2 + y^2 = 12$ were graphed in the standard (x, y) coordinate plane?

A circle **B** ellipse
C parabola **D** straight line
E 2 rays forming a "V"

3. If $999 \times 111 = 3 \times 3 \times n^2$, then which of the following could equal n?

A 9 **B** 37 **C** 111
D 222 **E** 333

4. In the figure below, $\triangle ABC$ is an equilateral triangle with $\overline{BC}$ 7 units long. If $\angle DCA$ is a right angle and $\angle D$ measures 45°, what is the length of $\overline{AD}$ in units?



A 7 **B** $7\sqrt{2}$ **C** 14 **D** $14\sqrt{2}$
E It cannot be determined from the information given.

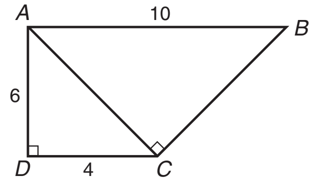
5. If $4 < a < 7 < b < 9$, then which of the following best defines $\frac{a}{b}$?

A $\frac{4}{9} < \frac{a}{b} < 1$ **B** $\frac{4}{9} < \frac{a}{b} < \frac{7}{9}$
C $\frac{4}{7} < \frac{a}{b} < \frac{7}{9}$ **D** $\frac{4}{7} < \frac{a}{b} < 1$
E $\frac{4}{7} < \frac{a}{b} < \frac{9}{7}$

6. A swimming pool with a capacity of 36,000 gallons originally contained 9,000 gallons of water. At 10:00 A.M. water begins to flow into the pool at a constant rate. If the pool is exactly three-fourths full at 1:00 P.M. on the same day and the water continues to flow at the same rate, what is the earliest time when the pool will be completely full?

A 1:40 P.M. **B** 2:00 P.M. **C** 2:30 P.M.
D 3:00 P.M. **E** 3:30 P.M.

7. In the figure below, what is the length of $\overline{BC}$?

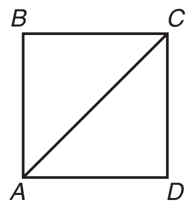


A 6 **B** $4\sqrt{3}$ **C** $2\sqrt{13}$
D 8 **E** $2\sqrt{38}$

8. If $\frac{x^2 + 7x + 12}{x + 4} = 5$, then $x =$

A 1 **B** 2 **C** 3 **D** 5 **E** 6

9. What is the value of $\frac{AC}{AD}$ if $ABCD$ is a square?



A 1
B $\sqrt{2}$
C $\sqrt{3}$
D 2
E $2\sqrt{2}$

10. Grid-In Segment AB is perpendicular to segment BD. Segment AB and segment CD bisect each other at point X. If $AB = 8$ and $CD = 10$, what is the length of $\overline{BD}$?

interNET CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**